Kolmogorov Complexity and Logical Formulas

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Trailer

$$K(x \to y) = K(y|x)$$

$$K(((x \to y) \to y) \to (x \vee y)) = ?$$

Outline

Introduction
- Motivation
- Definitions
- Illustrations

Case Studies

Complexity and Logic

Notation: programs

$$K(x) = \min\{\ell(p) \mid U(p) = x\}$$

U is a universal (or optimal) programming language

U is fixed, omit it:
$[p](x_1, \ldots, x_n)$ application of program p to inputs $x_1, \ldots, , x_n$

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Notation: equalities

$$f(n) = g(n) \quad (\text{or } f(n) \approx g(n)) \quad \text{iff } \exists C\, \forall n\, |f(n) - g(n)| \leq C \log n$$

Example

$K(\langle x, y\rangle) \approx K(x) + K(y|x)$ means
$\exists C\, \forall x, y \quad |K(\langle x, y\rangle) - (K(x) + K(y|x))| \leq C \log(\ell(x) + \ell(y))$

$K()$ plain or prefix complexity

$$K_{plain}(x) \approx K_{prefix}(x)$$

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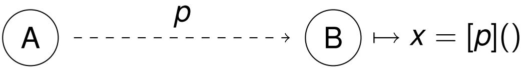
Complexity and Logic

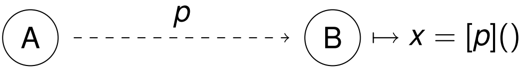
$K(x)$ and $K(x|y)$



$$\min \ell(p) = K(x)$$

$$\min \ell(p') = K(x|y)$$

$K(x)$ and $K(x|y)$



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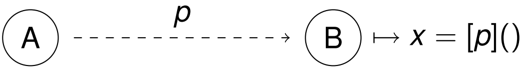

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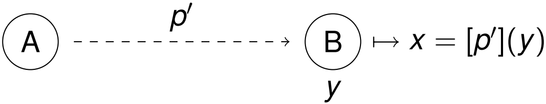

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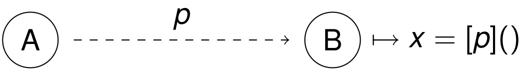
$K(x)$ and $K(x|y)$



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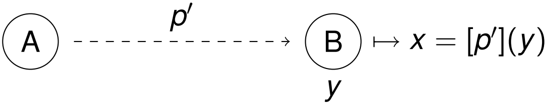
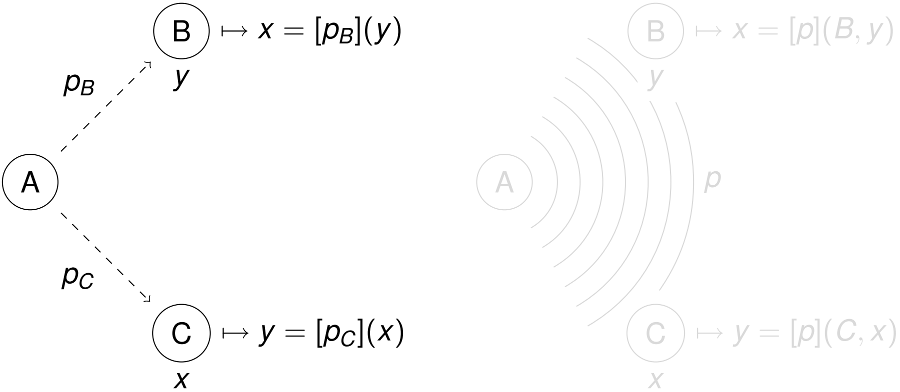


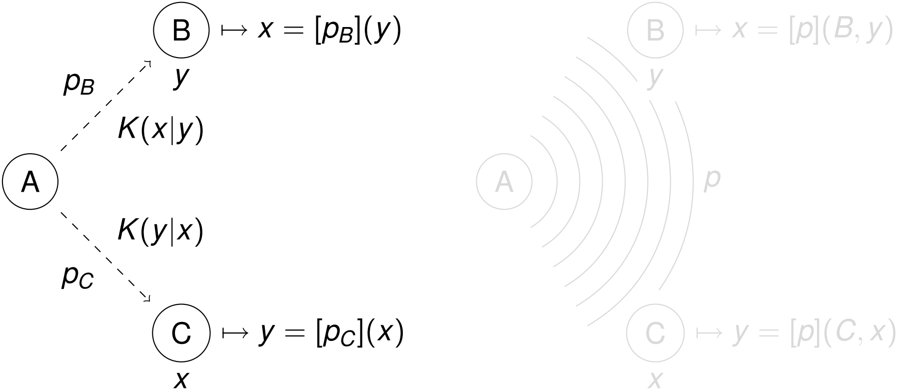
$$\min \ell(p') = K(x|y)$$

Information Distance



Bennett, Gács, Li, Vitányi, Zurek, 1993

Information Distance



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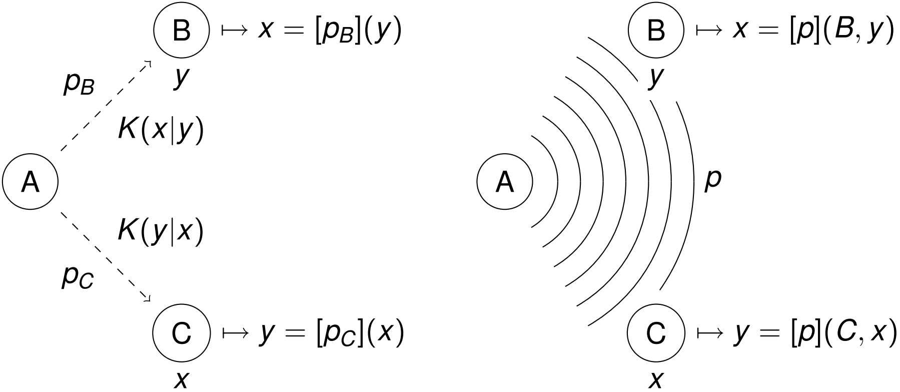
Information Distance



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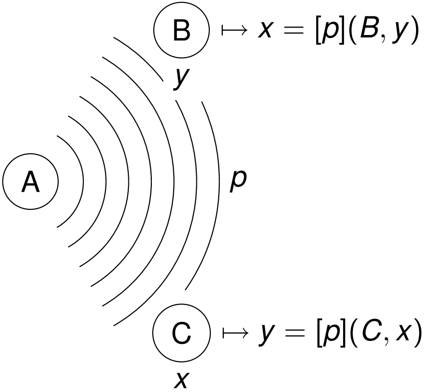
Information Distance

Information distance:

$$E(x, y) = \min \ell(p)$$

$$E(x, y) \geq K(x|y), K(y|x)$$
$$E(x, y) \leq K(x|y) + K(y|x)$$



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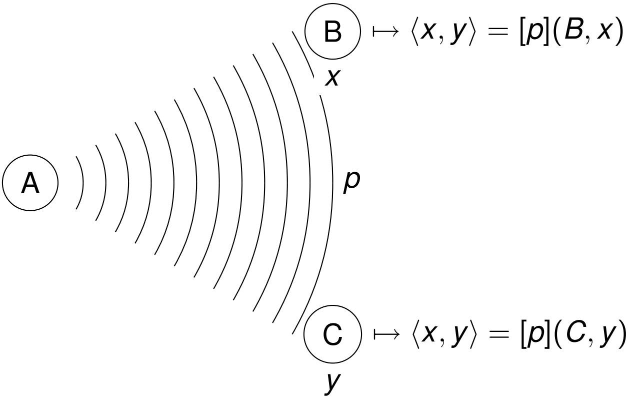
Multiconditional Complexity



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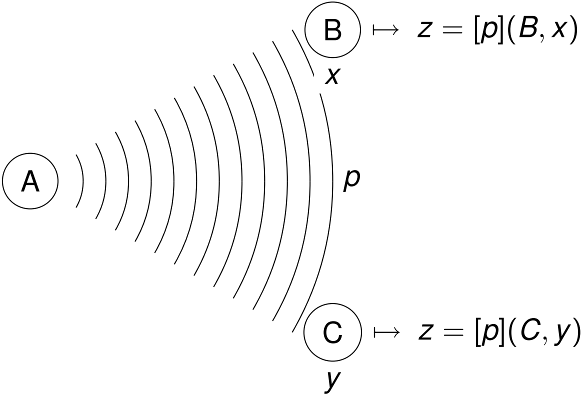
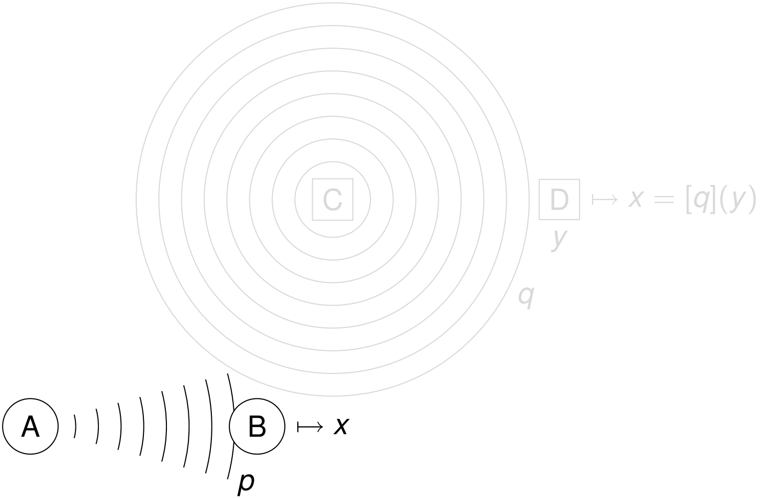


A Tricky Condition



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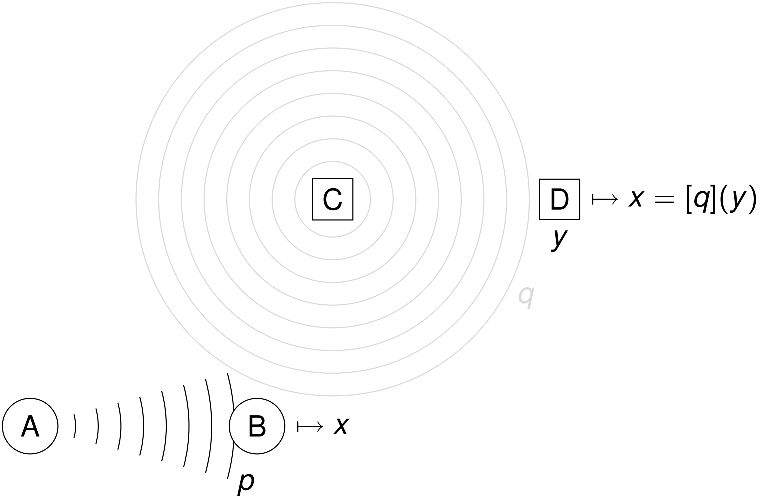

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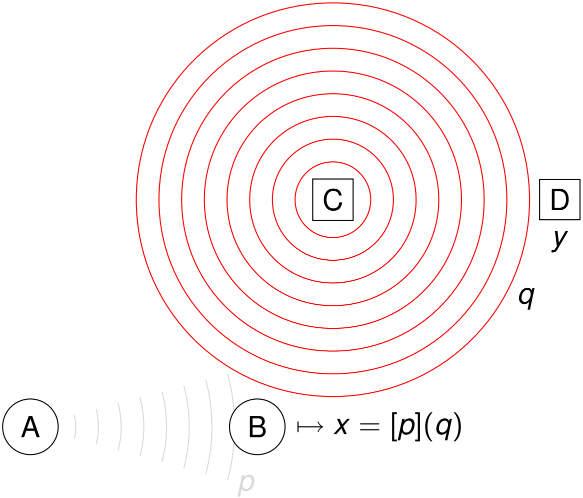
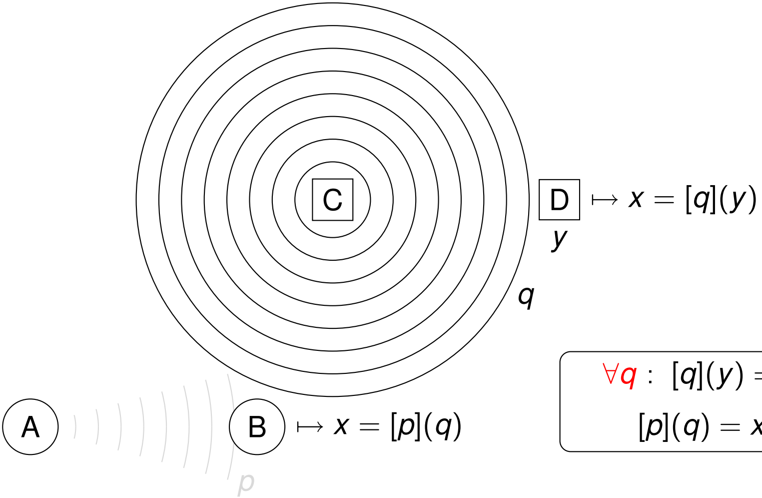


D $\mapsto x = [q](y)$

B $\mapsto x = [p](q)$

A Tricky Condition



$$\forall q : \; [q](y) = x$$

$$[p](q) = x$$

Definitions: $\to$

X, Y are sets (of constructive objects)

$$X \to Y = \{\, p \mid \forall x \in X \;\; [p](x) \in Y \,\}$$

Example (Conditional Complexity)

$K(x|y) = \min\{\, \ell(p) \mid [p](y) = x \,\}$

Example (A Tricky Condition)

$p : \;(\forall q : \;[q](y) = x) \quad [p](q) = x$

Definitions: $\rightarrow$

X, Y are sets (of constructive objects)

$$X \rightarrow Y = \{ p \mid \forall x \in X \ [p](x) \in Y \}$$

Example (Conditional Complexity)

$K(x|y) = \min\{ \ell(p) \mid [p](y) = x \}$

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$K(x|y) = \min\{\, \ell(p) \mid p \in \{y\} \to \{x\} \,\}$

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$p \in (\{y\} \to \{x\}) \to \{x\}$

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Example (A Tricky Condition)

$p \in (y \to x) \to x$

Definitions: $\to$ and $K()$

X, Y are sets (of constructive objects)

$$X \to Y = \{\, p \mid \forall x \in X\ \ [p](x) \in Y \,\}$$

$$K(X) = \min_{[p]() \in X} \ell(p) = \min_{x \in X} K(x)$$

Example (Conditional Complexity)

$K(x \to y) = \min\{\, \ell(p) \mid [p]() \in x \to y \,\} =$
$\min\{\, \ell(p) \mid [[p]()](x) = y \,\} \approx \min\{\, \ell(q) \mid [q](x) = y \,\} = K(y|x)$

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Definitions: Calculus of problems

X, Y are sets (of constructive objects)

$$X \to Y = \{\, p \mid \forall x \in X \;\; [p](x) \in Y \,\}$$
$$X \wedge Y = \{\, \langle x, y \rangle \mid x \in X,\ y \in Y \,\}$$
$$X \vee Y = \{\, \langle 0, x \rangle \mid x \in X \,\} \cup \{\, \langle 1, y \rangle \mid y \in Y \,\}$$

$$K(X) = \min_{x \in X} K(x)$$

Kolmogorov, 1932: Calculus of problems
Kleene, 1945: Arithmetic Realizability

Definitions: Calculus of problems

X, Y are sets (of constructive objects)

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Definitions: Calculus of problems

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Definitions: Calculus of problems

X, Y are sets (of constructive objects)

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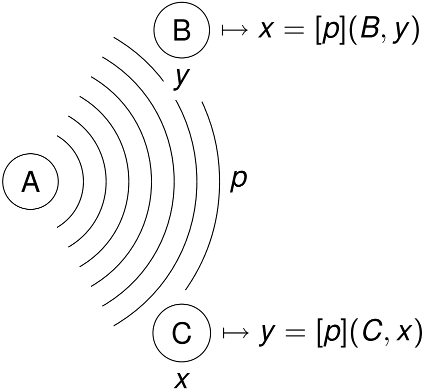
Information Distance

Information distance:

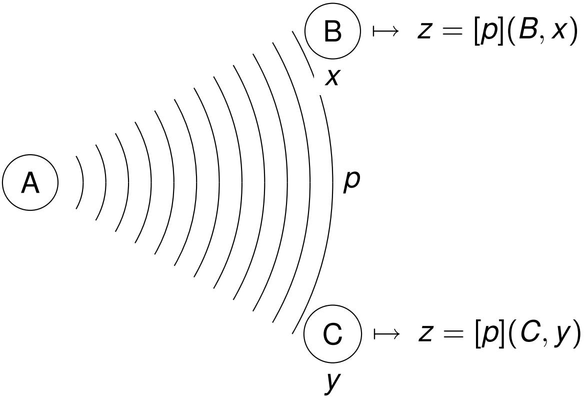
$$E(x, y) = \min \ell(p)$$



$$E(x, y) = K((x \to y) \wedge (y \to x))$$
$$E(x, y) = K((x \vee y) \to (x \wedge y))$$

Bennett, Gács, Li, Vitányi, Zurek, 1993

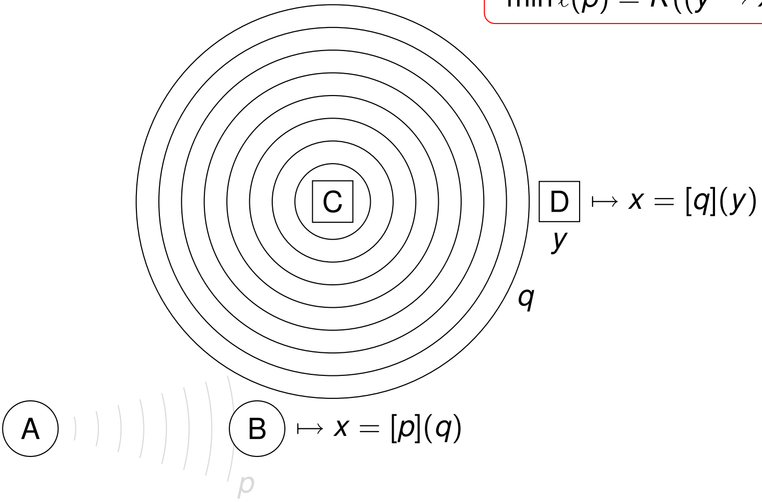
Multiconditional Complexity



$$\min \ell(p) = K(x \vee y \to z) = K((x \to z) \wedge (y \to z))$$

A Tricky Condition

$$\min \ell(p) = K((y \to x) \to x)$$



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Simplest Formulas

$$K(x \to y) = K(y|x)$$

$$K(x \wedge y) = K(\langle x, y \rangle)$$

$$K(x \vee y) =$$

$$= \min\{K(x), K(y)\}$$

Simplest Formulas

$$K(x \to y) = K(y|x)$$

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$$K(x \to y) = K(y|x)$$

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Simplest Formulas

$$K(x \to y) = K(y|x)$$

$$K(x \wedge y) = K(\langle x, y \rangle)$$

$$K(x \vee y) = K(\{\langle 0, x \rangle, \langle 1, y \rangle\}) = \min\{K(\langle 0, x \rangle), K(\langle 1, y \rangle)\}$$
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Simplest Formulas

$$K(x \to y) = K(y|x)$$

$$K(x \land y) = K(\langle x, y \rangle)$$

$$K(x \lor y) = K(\{\langle 0, x \rangle, \langle 1, y \rangle\}) = \min\{K(\langle 0, x \rangle), K(\langle 1, y \rangle)\}$$
$$= \min\{K(x), K(y)\}$$

Simplest Formulas

$$K(x \to y) = K(y|x)$$

$$K(x \wedge y) = K(\langle x, y\rangle)$$

$$\begin{aligned} K(x \vee y) &= K(\{\langle 0, x\rangle, \langle 1, y\rangle\}) = \min\{K(\langle 0, x\rangle), K(\langle 1, y\rangle)\} \\ &= \min\{K(x), K(y)\} \end{aligned}$$

Information Distance

$$E(x,y) = K((x \vee y) \to (x \wedge y)) = K((x \to y) \wedge (y \to x))$$

$$K((x \to y) \wedge (y \to x)) \geq \max\{K(x|y), K(y|x)\}$$

Information Distance

$$E(x,y) = K((x \vee y) \to (x \wedge y)) = K((x \to y) \wedge (y \to x))$$

$$K((x \to y) \wedge (y \to x)) \geq \max\{K(x|y), K(y|x)\}$$

Example

x, y are binary strings of length n,
random ($K(x) = K(y) = n$) and independent ($K(x|y) = K(y|x) = n$)
Then

$$K((x \to y) \wedge (y \to x)) = n = \max\{K(x|y), K(y|x)\}$$

Proof.

$$p = x \oplus y$$

□

Information Distance

$$E(x,y) = K((x \vee y) \to (x \wedge y)) = K((x \to y) \wedge (y \to x))$$

$$K((x \to y) \wedge (y \to x)) \geq \max\{K(x|y), K(y|x)\}$$

Theorem (Bennett, Gács, Li, Vitányi, Zurek, 1993)

$$E(x,y) = \max\{K(x|y), K(y|x)\}$$

More precisely,

$$E(x,y) - \max\{K(x|y), K(y|x)\} = O(\log(K(x|y) + K(y|x))).$$

Multiconditional Complexity

$$E(x, y) = K((x \vee y) \to z) = K((x \to z) \wedge (y \to z))$$

$$K((x \vee y) \to z) \geq \max\{K(z|x), K(z|y)\}$$

Multiconditional Complexity

$$E(x, y) = K((x \vee y) \to z) = K((x \to z) \wedge (y \to z))$$

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Theorem (An. Muchnik, 2000)

$$K((x \vee y) \to z) = \max\{K(z|x), K(z|y)\}$$

Multiconditional Complexity

$$E(x,y) = K((x \vee y) \to z) = K((x \to z) \wedge (y \to z))$$

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Theorem (An. Muchnik, 2000)

$$K((x \vee y) \to z) - \max\{K(z|x), K(z|y)\} = O(\log K(z))$$

Theorem (Gorbunov, 1998)

$\exists^\infty x, y, z$

$$K((x \vee y) \to z) = K(z|x) + K(z|y) + O(\log(K(z|x) + K(z|y)))$$

Multiconditional Complexity

$$E(x, y) = K((x \vee y) \to z) = K((x \to z) \wedge (y \to z))$$

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Theorem (An. Muchnik, 2000)

$$K((x \vee y) \to z) - \max\{K(z|x), K(z|y)\} = O(\log K(z))$$

Theorem (Gorbunov, 1998)

$\exists^\infty x, y, z$

$$K((x \vee y) \to z) = K(z|x) + K(z|y) + O(\log(K(z|x) + K(z|y)))$$

$$K(z) \gtrsim 2^{K(z|x)+K(z|y)}$$

More Examples

Theorem (Shen, Vereshchagin, 2001)

$$K((y \to x) \to x) = \min\{K(x), K(y)\} = K(x \vee y)$$

Theorem (Shen, Vereshchagin, 2001)

$$K(((y \to x) \to x) \to (x \vee y)) = \min\{K(x|y), K(y|x)\}$$

Theorem (Shen, Vereshchagin, 2001)

$$K(((y \to x) \to y) \to y) = 0$$

More Examples

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More Examples

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$$K((y \to x) \to x) = \min\{K(x), K(y)\} = K(x \vee y)$$

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More Examples

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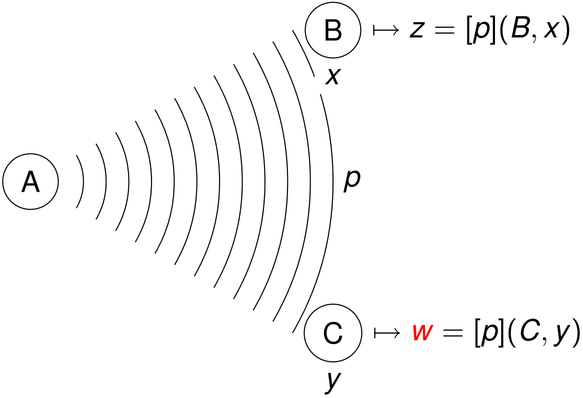
Non-reducible Complexity Formula



$$\min \ell(p) = K((x \to z) \wedge (y \to w))$$

Non-reducible Complexity Formula

Theorem (An. Muchnik, Vereshchagin, 2001)

$\forall n \, \exists \, \langle x_1, y_1, z_1, w_1 \rangle$ *and* $\langle x_2, y_2, z_2, w_2 \rangle$ *s.t.*

1. $K(x_i) = K(y_i) = K(z_i) = K(w_i) = n$,
   $K(x_i, y_i) = K(x_i, z_i) = \ldots = 2n$,
   $K(x_i, y_i, z_i) = K(x_i, z_i, w_i) = \ldots = 3n$, $K(x_i, y_i, z_i, w_i) = 3n$;
2. $$K((x_1 \to z_1) \wedge (y_1 \to w_1)) = 2n$$
3. $$K((x_2 \to z_2) \wedge (y_2 \to w_2)) = n$$

Outline

Introduction

Motivation

Definitions

Illustrations

Case Studies

Complexity and Logic

Complexity of Pair

$$K(\langle x, y\rangle) = K(x) + K(y|x)$$

Fact

For any X, Y

$$K(X \wedge Y) \leq K(X) + K(X \to Y)$$

$\forall n\, \exists X, Y$

$$K(X \wedge Y) = 2n, \quad K(X) = n, \quad K(X \to Y) = 2n$$

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Upper Bound for Complexity

$\Phi(A_1, \ldots, A_n)$ is a propositional formula with connectives $\wedge$, $\vee$, $\rightarrow$

Fact

For any sets $X_1, \ldots, X_n$

$$K(\Phi(X_1, \ldots, X_n)) \leq K(X_1 \wedge \ldots \wedge X_n)$$

Small Complexity Formulas

$\Phi(A_1, \ldots, A_n)$ is of small complexity iff
$\forall X_1, \ldots, X_n \quad K(\Phi(X_1, \ldots, X_n))$ is small

Example

$A \to A$ is of small complexity:

$\exists C\, \forall X \qquad K(X \to X) < C$

$K(\Phi \to \Psi)$ is of small complexity $\Rightarrow$ $K(\Psi) \leq K(\Phi)$

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Equalities and Tautologies

$$K((x \vee y) \to (x \wedge y)) = K((x \to y) \wedge (y \to x))$$

$$((A \vee B) \to (A \wedge B)) \leftrightarrow ((A \to B) \wedge (B \to A))$$

$$K((x \vee y) \to z) = K((x \to z) \wedge (y \to z))$$

$$((A \vee B) \to C) \leftrightarrow ((A \to C) \wedge (B \to C))$$

$$K(x \to (y \to z)) = K((x \wedge y) \to z)$$

$$(A \to (B \to C)) \leftrightarrow ((A \wedge B) \to C)$$

Logic of Small Complexity Formulas

Logic is a set of formulas closed under inference rules:

$$\frac{\Phi, \quad \Phi \to \Psi}{\Psi} \qquad \frac{\Phi(A_1, \ldots, A_n)}{\Phi(\Psi_1(B_1, \ldots, B_k), \ldots, \Psi_n(B_1, \ldots, B_k))}$$

$K(\Phi)$, $K(\Phi \to \Psi)$ are small $\Rightarrow$

$K(\Psi) \leq K(\Psi \wedge \Phi) \leq K(\Phi) + K(\Phi \to \Psi)$ is small

$K(\Phi(X_1, \ldots, X_n))$ is small for any $X_1, \ldots, X_n$ $\Rightarrow$

also for $X_i = \Psi_i(Y_1, \ldots, Y_k)$

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Intuitionistic formulas

$\mathfrak{Int}$ is the intuitionistic propositional logic

$\mathfrak{Int}$ is the logic with axioms

$A \to (B \to A), \quad (A \to (B \to C)) \to ((A \to B) \to (A \to C)),$
$(A \land B) \to A, \quad \dots, \quad (A \land \neg A) \to B.$

$(A \lor \neg A) \notin \mathfrak{Int} \qquad (\neg\neg A \to A) \notin \mathfrak{Int} \qquad (((A \to B) \to A) \to A) \notin \mathfrak{Int}$

$\Phi \in \mathfrak{Int} \quad \Rightarrow \quad \exists C\, \forall X_1, \dots, X_n \quad K(\Phi(X_1, \dots, X_n)) \leq C$

Fact

$\mathfrak{Int}$ is a logic of small complexity formulas

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Non-Small Complexity Formulas

Theorem (Chernov, Skvortsov, Skvortsova, Vereshchagin, 2002)

$\mathfrak{Int}$ *is the only logic of small complexity formulas*

Theorem (Chernov, 2003)

$\Phi \notin \mathfrak{Int} \quad \Rightarrow \quad \exists C \, \forall N$ *there are finite sets* $X_1, \ldots, X_n$

$$K(\Phi(X_1, \ldots, X_n)) \geq N - C, \qquad K(X_1 \wedge \ldots \wedge X_n) \leq N + C$$

$$K(\Phi(X_1, \ldots, X_n)) \leq K(X_1 \wedge \ldots \wedge X_n) + C$$

Non-Small Complexity Formulas

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Non-Small Complexity Formula: an Example

$$(((A \to B) \to A) \to A) \notin \mathfrak{Int}$$

$$X = \{x, y\}, \quad Y = \{y\}$$

$$\forall x, y \qquad K(((X \to Y) \to X) \to X) \geq \min\{K(x|y), K(y|x)\} + C$$

$$\forall x, y \qquad K(((x \to y) \to x) \to x) \leq O(\log(K(x) + K(y)))$$

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Lower Bounds: Proof Method

Example (Vereshchagin, 2001)

$$K((y \to x) \to x) \geq \min\{K(x), K(y)\} + O(\log(K(x) + K(y)))$$

Proof (part 1)

Given any $p \in ((y \to x) \to x)$, we construct z s.t.

$K(x|z) \leq const$ or $K(y|z) \leq const$

To this end, we run p on $q_1, \ldots, q_N$ s.t. $q_i \in (y \to x)$ for some i.

q_i are all functions from S^S, S is any finite set s.t. $x, y \in S$
e. g. $S = \{\, u \mid \ell(u) \leq \ell(x), \ell(y) \,\}$

$z = \langle u, v \rangle$ s.t. $(\forall q \;:\; q(u) = v) \quad p(q) = v$

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$p \in ((y \to x) \to x), \quad z = \langle u, v \rangle$ s.t. $(\forall q : q(u) = v) \quad p(q) = v$

1. $(\forall q : q(x) = y) \quad p(q) = y$
2. $(\forall q : q(u) = v) \quad p(q) = v$
   $u \neq x \quad \Rightarrow \quad \exists q : q(x) = y, \; q(u) = v \quad \Rightarrow \quad y = p(q) = v$

$$z = \langle x, v \rangle \quad \text{or} \quad z = \langle u, y \rangle$$

□

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Open Directions

- Formulas of singletons: general complexity properties
  - $K((x \to z) \wedge (y \to w))$ is non-reducible. Are there non-reducible formulas of two variables? Of three variables?
  - Classification of two-variable formulas, "when $K(\Phi(x, y) \to \Psi(x, y))$ is small?"
- Other operations on sets (tasks)
  - Appearing in proofs, as $X \tilde{\triangledown} Y = \{ \langle u, v \rangle \mid u \in X \text{ or } v \in Y \}$
  - Important for communication in multisource networks
- Applications to logical questions
  - Rose problem: constant realizability logic