# Definitional Extensions in Type Theory Revisited*

Georgiana Elena Lungu[1] and Zhaohui Luo[2]†

[1] Royal Holloway, Univ of London
Georgiana.Lungu.2013@live.rhul.ac.uk
[2] Royal Holloway, Univ of London
zhaohui.luo@hotmail.co.uk

The notion of definitionality or extension by definition was first formulated by Kleene [2] for first order theories. In first order logic we consider an extension obtained by adding a new symbol and an axiom that a formula involving the new symbol is equivalent to one in the base system. For an extension, conservativity means that all the formulas that are syntactically in the base system hold in the extension only if they hold in the base system. Definitionality, in addition, means that, an encompassing formula holds if and only if the corresponding formula holds if it is obtained by replacing the occurrences of the new formula with the equivalent ones from the base system.

We are interested in a similar notion for type theories like Martin Löf's type theory [6] or UTT [4]. We consider two different notions of definitional extension and, in particular, reformulate them in new ways, leading to a more generic notion of definitional extension that covers both as special cases. In the following, we use $LF$ to denote Martin-Löf's logical framework with labelled lambda-expressions (see Chapter 9 of [4]).

The first notion of definitional extension is related to the notion of conservativity as studied in [1, 3] where an embedding of a type theory into its extension is used and induces a particular notion of definitional extension with new symbols. In the setting of $LF$, it can be formulated as follows. (1) Let $T$ and $T'$ be type theories specified in $LF$ and $T'$ an extension of $T$ by adding new terms and rules. (2) Let $f$ be a mapping from the terms of $T'$ to those of $T$ such that (i) $f|_T = id_T$, and (ii) the new rules involving the new terms in $T'$ all become admissible under $f$ in $T$. Then $T'$ is a definitional extension of $T$ iff $T'$ is a conservative extension of $T$ and the definition rules of the form $\frac{\Gamma\vdash k:K \quad \Gamma\vdash f(k):K}{\Gamma\vdash k=f(k):K}$ are admissible in $T'$.[1] For example, we may consider $\Sigma$, a type theory with $\Sigma$-types, and $\Sigma[\times]$, the extension of $\Sigma$ with product types with expected rules. A syntactic map can be defined to map product types $A\times B$ to the $\Sigma$-type $\Sigma(A,[x{:}A]B)$ (and similarly for pairs and projections). Then, if in $\Sigma[\times]$ the definition rules such as $\frac{\Gamma\vdash A:Type \quad \Gamma\vdash B:Type}{\Gamma\vdash A\times B=\Sigma(A,[x:A]B)}$ are admissible, then $\Sigma[\times]$ is a definitional extension of $\Sigma$. In this setting, some meta-theoretic properties are carried over from $T$ to its definitional extension $T'$. For instance, if kind uniqueness holds for $T$, so does it for $T'$.

The above notion of definitional extension relies on the existence of a syntactic mapping and it does not cover some more general situations where, for example, the set of terms of the extension is the same as that of the extended calculus. It also does not deal with situations where new judgement forms are added. Coercive subtyping is such an example, with both of these features. It has been discussed by Xue et al. [7, 5, 8]. Instead of considering mapping between terms, they have described a definitional extension by mapping a derivation tree in the extension into one in the original calculus such that their conclusions are definitionally equal.[2]

*TYPES 2017.

†Supported by EU COST Action CA15123 and CAS/SAFEA International Partnership Program.

[1] Another way to think of this is that $T'$ extends $T$ with new terms and new rules, including those definition rules which correspond to the definition axiom in Kleene's setting of first-order theories.

[2] Note that Xue in [7] does not consider situations with new forms of judgements. Rather, he considers the calculus $T[C]_{0K}$ which has the same judgement forms with the whole calculus and is a conservative extension of the original calculus.

Here we propose a more general notion of definitional extension that has both notions above as special cases. Let $T'$ be an extension of $T$ with new terms, new rules and/or new forms of judgements. We define a notion of replacement of a judgement $J'$ in $T'$ by a judgement $J$ in $T$ based on a mapping $m : \mathcal{J}_{T'} \longrightarrow \mathcal{J}_T$ from $T'$-judgements to $T$-judgements so that $m$ is the identity when restricted to $T$-judgements ($m|_{\mathcal{J}_T} = id_{\mathcal{J}_T}$) and, when restricted to derivable judgements, respects definitional equality. Note that the mapping $m$ is now based on judgements, not on terms. This allows us to deal with the situation where the terms in $T$ and those in $T'$ are the same (as in coercive subtyping). For the first notion of definitional extension given above based on a syntactic mapping $f$, we can simply take $m$ to be the extension of $f$ to judgements. For the notion of definitional extension for coercive subtyping, as discussed in [5, 8], $m$ maps a judgement $\Gamma \vdash A \leq_c B$ to $\Gamma \vdash c : (A)B$. It is important to note that the mapping $m$ is syntactic and does not preserve derivability and, because of this, the definition of replacement is more complex and subtle in order to introduce and consider only derivable judgements (and we omit the details in the current abstract).

For an extension to be definitional, we require that it be conservative, and that any judgement in the extension have a replacement in the base calculus w.r.t. any of its derivation trees. This definition is similar to the one given in [7], in first place because even though a replacement is defined w.r.t. to a derivation, this definition refers to judgements as opposed to derivations, and it can cover new forms of judgements.

Note that, in order to discuss definitionality, it is important to understand what to replacement means, what a judgement can be replaced with, and how replacement should be done in a more general setting where, for example, there are new forms of judgements. In the previous settings, the answer to these questions was essentially covered by definition rules. Intuitively, one simply replaces a judgement with a definitionally equal one in a derivation tree to obtain a valid derivation tree (possibly with the addition to some more equality judgements). This cannot be the case for new forms of judgements or even for those extensions which add new forms of entries to the assumptions. These situation are all covered with our notion of definitional extension.

# References


[1] M. Hofmann. *Extensional concepts in intensional type theory.* PhD thesis, Univ of Edinburgh, 1995.

[2] S. Kleene. *Introduction to Metamathematics.* North Holland, 1952.

[3] P. LeFanu Lumsdaine. *Higher Categories from Type Theories.* PhD thesis, CMU, 2010.

[4] Z. Luo. *Computation and Reasoning: A Type Theory for Computer Science.* OUP, 1994.

[5] Z. Luo, S. Soloviev, and T. Xue. Coercive subtyping: theory and implementation. *Information and Computation*, 223:18–42, 2013.

[6] B. Nordström, K. Petersson, and J. Smith. *Programming in Martin-Löf's Type Theory: An Introduction.* Oxford University Press, 1990.

[7] T. Xue. Definitional extension in type theory. In R. Matthes and A. Schubert, editors, *LIPIcs Proceedings 19th International Conference on Types for Proofs and Programs*, 2013.

[8] T. Xue. *Theory and Implementation of Coercive Subtyping.* PhD thesis, Royal Holloway, University of London, 2013.