On n-partite tournaments with unique n-cycle

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Abstract

An n-partite tournament is an orientation of a complete n-partite graph. An n-partite tournament is a tournament, if it contains exactly one vertex in each partite set. Douglas, Proc. London Math. Soc. 21 (1970) 716-730, obtained a characterization of strongly connected tournaments with exactly one Hamilton cycle (i.e., n-cycle). For $n \geq 3$, we characterize strongly connected n-partite tournaments that are not tournaments with exactly one n-cycle. For $n \geq 5$, we enumerate such non-isomorphic n-partite tournaments.

1 Introduction

We use terminology and notation of [1]; all necessary notation and a large part of terminology used in this paper are provided in the next section.

The problems of characterizing strongly connected tournaments having a unique Hamilton cycle and finding the number of such tournaments have attracted several researches. R.J. Douglas [4] gave a structural characterization of tournaments having a unique Hamilton cycle. This result implies a formula for the number s_n of non-isomorphic tournaments of order n with a unique Hamilton cycle. This characterization as well as the formula are rather complicated. G.P. Egorychev [5] obtained a simplification by considering a generating function for the number of tournaments with unique Hamiltonian cycle. M.R. Garey [6] showed that s_n could be expressed as a Fibonacci number ($s_n = f_{2n-6}$); his derivation was based on Douglas's characterization. J.W. Moon [13] obtained a direct proof of Garey's formula that is essentially independent of Douglas's characterization.

A classical result of J.A. Bondy [3] on short cycles in strong n-partite tournaments implies that every strong n-partite tournament contains an n-cycle.

In this paper, we provide a characterization of strongly connected n-partite tournaments ($n \geq 3$) that are not tournaments with a unique n-cycle. We show that our characterization is polynomial time verifiable. The characterization allows us to find easily the number of such non-isomorphic n-partite tournaments for $n \geq 5$. Thus, our work is a natural continuation of the papers mentioned in the previous paragraph.

Our result is also related to a question of L. Volkmann [14]: let $n \geq 4$, is there a strongly connected n-partite tournament with at least two vertices in some partite set and with exactly $n - m + 1$ cycles of length m for each fixed $m \in \{4, 5, \ldots, n\}$? The first two authors [12] gave an affirmative answer to this question and raised a natural problem of characterizing such n-partite tournaments. Our paper solves the problem for $n = m$, which seems the most interesting case.

A recent paper [14] of L. Volkmann is the latest survey on cycles in multipartite tournaments. Cycles in multipartite tournaments were earlier overviewed in [2, 8, 10].

2 Terminology and Notation

A digraph obtained from an undirected graph G by replacing every edge of G with a directed edge (arc) with the same end-vertices is called an *orientation* of G. An *oriented graph* is an orientation of some undirected graph. A *tournament* is an orientation of a complete graph, and an *n-partite tournament* is an orientation of a complete n-partite graph. Partite sets of complete multipartite graphs become *partite sets* of n-partite tournaments.

The terms *cycle* and *path* mean simple directed cycle and path. A cycle of length k is a *k-cycle*. For a cycle $C = v_1v_2 \ldots v_kv_1$, $C[v_i, v_j]$ denotes the path $v_iv_{i+1} \ldots v_j$ which is part of C. A digraph D is *strong* (or, *strongly connected*) if for every ordered pair x, y of vertices in D there exists a path from x to y. For a set X of vertices of a digraph D, $D\langle X\rangle$ denotes the subdigraph of D induced by X.

For sets T, S of vertices of a digraph $D = (V, A)$, $T{\rightarrow}S$ means that for every vertex $t \in T$ and for every vertex $s \in S$, we have $ts \in A$, and $T{\Rightarrow}S$ means that for no pair $s \in S$, $t \in T$, we have $st \in A$. While for oriented graphs $T{\rightarrow}S$ implies $T{\Rightarrow}S$, this is not always true for general digraphs. If $u{\rightarrow}v$ (i.e., $uv \in A$), we say that u *dominates* v and v *is dominated* by u.

For a digraph $D = (V, A)$ and a vertex $x \in V$, $N^+(x) = \{y \in V : \ xy \in A\}$. Similarly, $N^-(x) = \{y \in V : \ yx \in A\}$. We denote, by $\mathcal{UC}_n$, the set of all strong n-partite tournaments, $n \geq 3$, which are not themselves tournaments, with exactly one cycle of length n.

We call $X = X_1 \cup X_2 \cup \ldots \cup X_s$ a *partition* of X and denote it by $X = X_1 \uplus X_2 \uplus \ldots \uplus X_s$, if $X_i \cap X_j = \emptyset$ for each $i \neq j$.

3 The case of $n \geq 4$

We start from two known results.

Theorem 3.1 *[9] Every partite set of a strong n-partite tournament, $n \geq 3$, contains a vertex which lies on an m-cycle for each $m \in \{3, 4, \ldots, n\}$.*

Theorem 3.2 *[7] Every vertex in a strong n-partite tournament, $n \geq 3$, belongs to a cycle that contains vertices from exactly q partite sets for each $q \in \{3, 4, \ldots, n\}$.*

Lemma 3.3 *Let D be a digraph with a cycle C. If for some vertices $x, y \in V(D) - V(C)$ we have $|N^+(x) \cap V(C)| + |N^-(y) \cap V(C)| > |V(C)|$, then there is an (x, y)-path of length l in D, for all $l = 2, 3, \ldots, |V(C)| + 1$.*

Proof: Suppose that there was no (x, y)-path of length l in D. Let $z \in N^-(y) \cap V(C)$ and let u be the vertex on C such that the path from u to z in C has length $l - 2$. Observe that u does not dominate x. Thus, $|N^+(x) \cap V(C)| \leq |V(C)| - |N^-(y) \cap V(C)|$, a contradiction. □

Lemma 3.4 *If $n \geq 3$, $D \in \mathcal{UC}_n$ and C is the unique n-cycle in D, then C contains vertices from all partite sets. There is a vertex $y \in D - V(C)$ such that $D\langle V(C) \cup y\rangle$ is strong.*

Proof: Let $D \in \mathcal{UC}_n$ and let C be its unique n-cycle. If C contains vertices from less than n partite sets, then by Theorem 3.1 there exists another n-cycle, which is impossible. Therefore C contains a vertex from every partite set of D. Let $V_1, V_2, \ldots, V_n$ be partite sets of D and let $C = v_1 v_2 \ldots v_n v_1$, $v_i \in V_i$, $i = 1, 2, \ldots, n$.

Assume that there is no vertex $y \in D - V(C)$ for which $D\langle V(C) \cup y\rangle$ is strong. Then the following two sets S and T are non-empty: S (T) is the set of vertices in $D - V(C)$ that do not dominate (are not dominated by) any vertex in C. Since D is strong, there exist vertices $u \in S$ and $v \in T$ such that $u \rightarrow v$. By Lemma 3.3 we obtain an n-cycle containing u and v, a contradiction. □

Lemma 3.5 *Let $n \geq 4$, let $D \in \mathcal{UC}_n$ and let C be the unique n-cycle in D. We may order the partite sets in D such that $C = v_1 v_2 \ldots v_n v_1$, where $v_i \in V_i$ and $v_i \rightarrow v_j$ for all $1 < j + 1 < i \leq n$. If $y \in V(D) - V(C)$ such that $D\langle V(C) \cup y\rangle$ is strong, then one of the following statements hold.*

(i): *$y \in V_2$, $y \rightarrow (\{v_1\} \cup \{v_3, v_4, \ldots, v_{n-1}\})$ and $v_n \rightarrow y$.*

(ii): *$y \in V_{n-1}$, $(\{v_n\} \cup \{v_{n-2}, v_{n-3}, \ldots, v_2\}) \rightarrow y$ and $y \rightarrow v_1$.*

Proof: By Lemma 3.4 $D\langle V(C)\rangle$ is a tournament and there exists a vertex $y \in V(D) - V(C)$ such that $D\langle V(C) \cup y\rangle$ is strong. Let $C = w_1 w_2 \ldots w_n w_1$ and assume that y belongs to the same partite set as w_i. By Theorem 3.2, y lies in a cycle C' of $D\langle V(C) \cup y\rangle$ that contains vertices from exactly $n-1$ partite sets. If C' contains w_i, then the length of C' is n, a contradiction. Thus, C' does not contain w_i and a vertex w_j, $j \neq i$. Let $T = D\langle V(C') \cup w_j\rangle$; T is a tournament. If T is strong, then it has a Hamilton cycle, which is an n-cycle in D distinct from C, a contradiction. Therefore we may assume that T is not strong.

Thus, either $C' \rightarrow w_j$ or $w_j \rightarrow C'$. First assume that $C' \rightarrow w_j$ (we will show that this implies (i) above). Since every vertex in C dominates a vertex, $C' \rightarrow w_j$ implies $w_j = w_{i-1}$. Let $C' = z_1 z_2 \ldots z_{n-1} z_1$ where $z_{n-1} = y$. Note that if $w_i \rightarrow z_r$, for any $r > 1$, then $w_i C'[z_r, z_{r-2}] w_{i-1} w_i$ is an n-cycle containing $y = z_{n-1}$, and thus distinct from C, a contradiction. Therefore $z_1 = w_{i+1}$ and $V(C') - \{w_{i+1}\} \Rightarrow w_i$.

We will now show that $z_q \rightarrow z_k$ for all $1 < k+1 < q \leq n-1$. To prove this claim, assume that $z_k \rightarrow z_q$ for some $1 < k+1 < q \leq n-1$. If $q = k+2$, then the cycle $w_i C'[z_1, z_{q-2}] C'[z_q, z_{n-1}] w_{i-1} w_i$ gives us a contradiction. Assume that q is maximum, and subject to that k is maximum. Then the cycle $w_i C'[z_1, z_k] C'[z_q, z_{n-1}] C'[z_{k+2}, z_{q-1}] w_{i-1} w_i$ gives us a contradiction. Since $C = w_i C'[z_1, z_{n-2}] w_{i-1} w_i$, we have proved the first part of the lemma by letting $v_1 = w_{i-1}$, $v_2 = w_i$, ..., $v_n = w_{i-2}$. In order to prove (i), we observe that $y \in V_2$ follows immediately from the fact that y and $w_i = v_2$ lie in the same partite set. The facts that $y \rightarrow \{v_1, v_3, v_4, \ldots, v_{n-1}\}$ and $v_n \rightarrow y$ also follow from the above as $y = z_{n-1}$, which dominates w_{i-1} and all z_k with $1 \leq k \leq n-3$.

The case when $w_j \rightarrow C'$ can be considered analogously to the above and, in particular, implies (ii). □

In what follows, $D \in \mathcal{UC}_n$ and $C = v_1 v_2 \ldots v_n v_1$ is the unique n-cycle of D. Assume that the ordering of the partite sets satisfies the conditions of Lemma 3.5.

Let us partition the vertices of $V_i - v_i$ $(1 \leq i \leq n)$ into four classes. We say that $x \in V_i - \{v_i\}$ is of *type* 1 if it satisfies condition (i) of Lemma 3.5, x is of *type* 2 if it satisfies condition (ii) of Lemma 3.5, and x is of *type* 3 if $V(C) \Rightarrow x$ and finally x is of *type* 4 if $x \Rightarrow V(C)$. Let T_i denote the set of vertices of type i $(1 \leq i \leq 4)$ in $D - V(C)$.

Remark. From the last lemma, we have that the vertices of type 1 are all in one partite set, V_2, the vertices of type 2 are all in the same partite set, V_{n-1}.

Lemma 3.6 *If $n \geq 4$, then $(T_1 \cup T_4) \Rightarrow (T_2 \cup T_3)$.*

Proof: If there is an arc uw from $T_2 \cup T_3$ to $T_1 \cup T_4$, then we observe that $|N^+(w) \cap V(C)| + |N^-(u) \cap V(C)| \geq (n-2) + (n-2) \geq n$. We are now done by Lemma 3.3, unless equality holds, in which case $n = 4$, $u \in T_2$ and $w \in T_1$. However in this case the 4-cycle uwv_1v_2u, gives us the desired contradiction. □

Remark. It is possible that T_1 or T_2 is empty, but by Lemmas 3.4 and 3.5 at least one of them is non-empty.

Lemma 3.7 *Let $n \geq 4$. Let $R_0 = T_1$ and $Q_0 = T_2$, and define R_i and Q_i, $i \geq 1$, as follows.*

$R_i = \{r \in T_4 :$ *the shortest path from R_0 to r in $D - V(C) - T_2 - T_3$ has length i*$\}$.

$Q_i = \{q \in T_3 :$ *the shortest path from q to Q_0 in $D - V(C) - T_1 - T_4$ has length i*$\}$.

Note that $R_j \Rightarrow R_i$ and $Q_i \Rightarrow Q_j$ for all $j > i+1 \geq 1$, by definition. The following must also hold for $i \geq 0$.

(a) $R_i \subseteq V_{(i \ mod \ n-1)+2}$

(b) $Q_i \subseteq V_{(n-2-i \ mod \ n-1)+1}$

Proof: We start of by proving (a) by induction on i. The claim (a) is clearly true for $i = 0$, by Lemma 3.5, so we may assume that $i > 0$. Also assume that $i < n-1$. If there exists a vertex $r_i \in R_i - V_{i+2}$, then let P be a path of length i from $r_0 \in R_0$ to r_i and observe that $PC[v_{i+2}, v_n]r_0$ is an n-cycle, a contradiction. Therefore $R_i \subseteq V_{i+2}$ when $i < n-1$.

If $i \geq n-1$, then $R_i \Rightarrow R_{i-(n-1)}$ by the definition of sets R_i. This would give us an n-cycle different from C, unless all vertices in R_i belong to the same partite set as all vertices in $R_{i-(n-1)}$. We are now done by induction hypothesis as $R_{i-(n-1)} \subseteq V_{((i-(n-1) \bmod n-1)+2} = V_{(i \bmod n-1)+2}$.

(b) can be proved analogously to (a). □

Lemma 3.8 *Consider R_i and Q_i as defined in Lemma 3.7. The following now holds.*

(a) If $n \geq 5$ then $R_i \rightarrow R_{i+1}$ and $Q_{i+1} \rightarrow Q_i$ for all $i \geq 0$.

(b) If $n \geq 6$ and $|R_i| > 0$ then $|R_{i-2}| = 1$ for all $i \geq 2$.

Furthermore if $|Q_i| > 0$ then $|Q_{i-2}| = 1$ for all $i \geq 2$.

Proof: We first prove the first part of (a). Assume that it is not true and that there exists $r_i \in R_i$ and $r_{i+1} \in R_{i+1}$ with $r_{i+1} \rightarrow r_i$. Let $P = p_0p_1 \ldots p_ir_{i+1}$ be a path from R_0 to r_{i+1} in $D - V(C) - T_2 - T_3$ of length $i+1$. Note that $r_i \notin P$. If $i+3 \geq n$, then we obtain the n-cycle $r_{i+1}r_iP[p_{i-n+3}, r_{i+1}]$, a contradiction (as $r_i \rightarrow p_{i-n+3}$, by Lemma 3.7 and the fact that $i-n+3 \leq i-2$). If $i+3 < n$ then we obtain the n-cycle $r_{i+1}r_iC[v_{i+4}, v_n]P$, a contradiction. This proves the first part of (a).

The second part of (a) can be proved analogously.

We now prove the first part of (b). Assume that it is not true. Let $r_i \in R_i$ be arbitrary and let $P = p_0p_1 \ldots p_{i-1}r_i$ be any path from R_0 to r_i in $D-V(C)-T_2-T_3$ of length i. Let $r_{i-2} \in R_{i-2}-p_{i-2}$ be arbitrary. If $i+2 \geq n$, then we obtain the n-cycle $r_ir_{i-2}P[p_{i-n+2}, r_i]$, a contradiction (as $r_{i-2} \rightarrow p_{i-n+2}$, by Lemma 3.7 and the fact that $i-n+2 \leq i-4$). If $i+2 < n$ then we obtain the n-cycle $r_ir_{i-2}C[v_{i+3}, v_n]P$, a contradiction. This proves (b).

The second part of (b) is proved analogously.

$\square$

Theorem 3.9 *Let $n \geq 4$ and let $Z = \{v_1, v_2, v_3, \ldots, v_n\}$. Let $R_0, R_1, \ldots R_a$ and $Q_0, Q_1, \ldots, Q_b$ be disjoint non-empty sets (if $a = -1$ then there are no R-sets and if $b = -1$ then there are no Q-sets). Let $R = R_0 \uplus R_1 \uplus \ldots \uplus R_a$ and $Q = Q_0 \uplus Q_1 \uplus \ldots \uplus Q_b$.*

Let D be a multipartite tournament with $V(D) = Z \uplus R \uplus Q$. Let $V_1, V_2, \ldots, V_n$ be the partite sets of D, where $v_i \in V_i$, $R_i \subseteq V_{(i \bmod n-1)+2}$ and $Q_i \subseteq V_{(n-2-i \bmod n-1)+1}$ for all i. Furthermore assume that the following statements hold.

(a) *For all $i = 1, 2, \ldots, n-1$ we have $v_i \rightarrow v_{i+1}$.*

(b) *For all $1 \leq j < i-1 < n$ we have $v_i \rightarrow v_j$.*

(c) *For all $0 \leq j < i-1 < a$ we have $R_i \Rightarrow R_j$, and for all $0 \leq j < i-1 < b$ we have $Q_j \Rightarrow Q_i$.*

(d) *We have $R \Rightarrow Z \Rightarrow Q$ and $R \Rightarrow Q$, except for $Q_0 \rightarrow v_1$ and $v_n \rightarrow R_0$.*

(e) *If $n \geq 6$, then $|R_0| = |R_1| = \ldots = |R_{a-2}| = 1$ and $|Q_0| = |Q_1| = \ldots = |Q_{b-2}| = 1$.*

(f) *If $n \geq 5$, then $R_i \rightarrow R_{i+1}$ for all $i = 0, 1, 2, \ldots, a-1$, and $Q_{i+1} \rightarrow Q_i$ for all $i = 0, 1, 2, \ldots, b-1$.*

(g) *If $n = 4$, then the following holds. Every vertex in R_{i+1} has an arc into it from R_i, for all $i = 0, 1, \ldots, a-1$. Every vertex in Q_{i+1} has an arc into Q_i, for all $i = 0, 1, \ldots, b-1$. Furthermore there is no 4-cycle in $D\langle R\rangle$ or $D\langle Q\rangle$.*

Then D contains a unique n-cycle. Any n-partite tournament, which is not a tournament, with a unique n-cycle, can be constructed as above.

Proof: We note that if D is a n-partite tournament, which is not a tournament, containing a unique n-cycle, then all of the above must hold by the previous lemmas (note that (g) follows trivially).

Now we prove that if D satisfies the above conditions then $C = v_1v_2v_3\dots v_nv_1$ is a unique n-cycle in D. Assume that C' is another n-cycle. By (a) and (b), C' is not completely in $D\langle Z\rangle$. We claim that $D\langle R\rangle$ and $D\langle Q\rangle$ do not contain an n-cycle. This follows from (g) for $n = 4$, from (c), (e), (f) and the fact that two R-sets or Q-sets with indices n apart are in the same partite set for $n \geq 5$ (for $n \geq 6$ (e) is important to prevent an n-cycle visiting the same R-set or Q-set twice). As $R \Rightarrow Q$, this implies that there is no n-cycle in $D\langle R \cup Q\rangle$. This implies that C' must contain at least one vertex from Z. We now consider the following cases.

Case 1: Assume that $V(C') \cap R \neq \emptyset$ and $V(C') \cap Q \neq \emptyset$. By (d), C' must contain v_n followed by vertices in R, then by vertices in Q and then the vertex v_1. However then we must have a path from v_1 to v_n in $D\langle Z\rangle$, and the only such path has length n, a contradiction to the length of C'.

Case 2: Assume that $V(C') \cap R \neq \emptyset$ and $V(C') \cap Q = \emptyset$. As before C' must contain v_n followed by vertices in R, then by vertices in Z, ending in v_n.

Let $r = |R \cap V(C')|$. If $n = 4$ we get a contradiction by considering the cases $r = 1, 2, 3$ separately. In particular, for $r = 3$, the path $C' \cap R$ has both initial and terminal vertices in R_0 or the initial vertex in R_0 and the terminal one in R_2. However, $v_4 \rightarrow R_0$, and v_4 and R_2 are in the same partite set.

If $n = 5$ and $r = 4$, then the predecessor of v_n on C' must lie in either R_0 or R_3, a contradiction. If $n \geq 6$, or $r \leq 3$ and $n = 5$, then the only path with r vertices in $D\langle R\rangle$ starting at R_0, has the form $s_0s_1s_2\dots s_{r-1}$, where $s_i \in R_i$. Since the only path into v_n with $n - r$ vertices in $D\langle Z\rangle$ is $v_{r+1}v_{r+2}\dots v_n$, we get a contradiction (s_{r-1} and v_{r+1} lie in the same partite set).

Case 3: Assume that $V(C') \cap R = \emptyset$ and $V(C') \cap Q \neq \emptyset$. We can obtain a contradiction analogously to Case 2. □

Corollary 3.10 *We can check whether a strong n-partite tournament, $n \geq 4$, belongs to $\mathcal{UC}_n$ in polynomial time.*

Proof: Let D be a strong n-partite tournament, $n \geq 4$. The above mentioned result of Bondy [3] implies that D has an n-cycle. The constructive proof in [3] implies a polynomial algorithm to construct such a cycle C. If C has two or more vertices from the same partite set, then by Theorem 3.1 $D \notin \mathcal{UC}_n$. Otherwise, we have to check the conditions of the above theorem that can be done in polynomial time. □

Corollary 3.11 *The number of non-isomorphic strong n-partite tournaments, $n \geq 6$,* *with $n+q$ vertices belonging to $\mathcal{UC}_n$ is*

$$t(n,q) = 2 + (q-1)q + \sum_{j=1}^{q-1}(1 + j(j-1)/2)(1 + (q-j)(q-j-1)/2).$$

The number of non-isomorphic strong 5-partite tournaments, with $q+5$ vertices belonging to $\mathcal{UC}_5$ is $t(5,q) = (q+3)2^{q-2}$.

Proof: It follows from Theorem 3.9 that $t(n,q)$, for $n \geq 5$, equals the number of possible partitions of sets R and Q (or of the non-empty one, if one of them is empty).

Let $n \geq 6$. If $R = \emptyset$, then q vertices of Q may all be in Q_0 or exactly one vertex in each of $Q_0, Q_1, \dots, Q_{b-2}$ and $q-b+1$ vertices are in $Q_{b-1} \cup Q_b$. Since there are exactly $q-b$ ways to distribute $q-b+1$ vertices among non-empty Q_{b-1} and Q_b and $1 \leq b \leq q-1$, the overall number of possible partitions of Q equals $1 + (q-1)q/2$.

To compute $t(n,q)$ for $n \geq 6$, it suffice to observe that the number of possible partitions of R also equals $1 + (q-1)q/2$, if $Q = \emptyset$, and if both R and Q are non-empty, then the number of possible partitions of R and Q equals $(1 + j(j-1)/2)(1 + (q-j)(q-j-1)/2)$ provided $|R| = j$.

Let $n = 5$. If $R = \emptyset$, then q vertices of Q may be distributed among the Q-sets as follows: we "open" Q_0 and put the first vertex in Q_0, the second vertex goes into Q_0 or into Q_1, but if it goes into Q_1, we "close" Q_0 and "open" Q_1 (similarly to the first fit heuristic in bin packing), etc. The kth vertex goes into the open Q_j or we "close" Q_j and "open" Q_{j+1} and put the vertex there. This algorithm implies that the number of possible partitions of Q equals 2^{q-1}.

Similarly to the case of $n \geq 6$, we see that $t(5,q)$ equals $2 \times 2^{q-1} + \sum_{j=1}^{q-1} 2^{j-1}2^{q-j-1}$. $\square$

Remark. It is interesting to observe that while $t(n,q)$ is polynomial for $n \geq 6$, the number is exponential, in q, for $n = 5$. Another interesting observation is that we have been able to find the exact number of certain *non-isomorphic* n-partite tournaments. For non-isomorphic acyclic and close to acyclic families of n-partite tournaments, an exact enumeration was given in [11].

4 The case of $n = 3$

Theorem 4.1 *A strong 3-partite tournament D with partite sets V_1, V_2 and V_3 contains a unique 3-cycle if and only if it can be constructed in one of the following two ways.*

(i) *A partite set, say V_1, has only one vertex x and there is exactly one arc from $N^+(x)$ to $N^-(x)$.*

(ii) *Each partite set has at least two vertices. Let* $v_1 \in V_1$*,* $v_2 \in V_2$ *and* $V_3 = \{v_3\} \uplus W_1 \uplus W_2$ *(*W_1 *or* W_2 *may be empty). Let* D *contain arcs such that* $(V_1 \cup W_1) \Rightarrow (V_2 \cup W_2)$ *and* $((V_1 \cup v_2) - v_1) \rightarrow v_3 \rightarrow ((V_2 \cup v_1) - v_2)$*. The arcs between* W_1 *and* V_1 *as well as between* W_2 *and* V_2 *are arbitrary.*

Proof: It is not difficult to check that a digraph constructed as in (i) above has a unique 3-cycle. Now let D be a digraph constructed as in (ii) above, and let $S_1 = W_1 \cup V_1$ and $S_2 = W_2 \cup V_2$. Observe that $D\langle S_i \cup \{v_3\}\rangle$ is bipartite for $i = 1, 2$. Therefore any 3-cycle must contain vertices from both S_1 and S_2. However, the only path from S_2 to S_1 is $v_2 v_3 v_1$, which gives us the unique 3-cycle $v_1 v_2 v_3 v_1$.

Now assume that D is a strong 3-partite tournament containing a unique 3-cycle. Let V_1, V_2 and V_3 be partite sets of D. We first consider the case when some partite set is of size 1. Without loss of generality assume that $V_1 = \{x\}$. Clearly, every arc from $N^+(x)$ to $N^-(x)$ results in a 3-cycle, and there must be at least one such arc as D is strong. This implies that D can be constructed as in (i) above. So now consider the case when all partite sets contain at least two vertices.

Let $C = v_1 v_2 v_3 v_1$ be the unique 3-cycle in D, with v_i belonging to the partite set V_i. Note that if $X \subseteq V(C)$ is non-empty then $D - X$ is not strong, by Theorem 3.1. Let $Q_1, Q_2, \ldots Q_s$ be the strong components of $D - V(C)$, such that $Q_i \Rightarrow Q_j$ for all $1 \leq i < j \leq s$. Observe that each Q_i has vertices from at most two partite sets. Let Q^* contain all vertices in $D - V(C)$ which have a path into it from Q_1 and a path from it to Q_s. The set $Q^* \neq \emptyset$ since $Q_1 \cup Q_s$ are in Q^*.

Observe that Q^* contains vertices from all three partite sets. As D is strong, there is an arc from C to Q_1. Without loss of generality assume that $v_1 q_1$ is such an arc. There is also an arc from Q_s to C. Note that this arc must enter v_2, as if it enters v_1 then $D\langle Q^* \cup \{v_1\}\rangle$ is strong and if it enters v_3 then $D\langle Q^* \cup \{v_1, v_3\}\rangle$ is strong. Let $q_s v_2$ be such an arc. By an analogous argument $q_1 \Rightarrow V(C) - v_1$ and $V(C) - v_2 \Rightarrow q_s$. This implies that q_1 and q_s belong to the partite set V_3, as we otherwise would obtain the 3-cycles $v_1 q_1 v_3 v_1$ and/or $q_s v_2 v_3 q_s$.

Now assume that there exists an arc $r_2 r_1$ in $D - V(C)$, where $r_2 \in V_2$ and $r_1 \in V_1$. If $r_2 \rightarrow v_3$, then let P be any path from q_1 to r_2 in $D - V(C)$ (which exists as $q_1 \in Q_1 \cap V_3$). Thus, $D\langle V(P) \cup \{v_3, v_1\}\rangle$ is strong and contains vertices from three partite sets. By Theorem 3.1, it contains a 3-cycle, which does not include v_2, a contradiction. Therefore $v_3 \rightarrow r_2$. We can analogously prove that $r_1 \rightarrow v_3$. However, this gives us the 3-cycle $v_3 r_2 r_1 v_3$, a contradiction. Therefore $(V_1 - v_1) \rightarrow (V_2 - v_2)$.

This means, in particular, that there exists an index $i \geq 1$, such that $V_1 - v_1 \subseteq Q_1 \cup Q_2 \cup \ldots \cup Q_i$ and $V_2 - v_2 \subseteq Q_{i+1} \cup Q_{i+2} \cup \ldots \cup Q_s$ (as all Q's can contain vertices from at most two partite sets). Furthermore there is no $r_2 \in V_2$, such that $r_2 \rightarrow v_1$ as we could then obtain the 3-cycle $v_1 q_1 r_2 v_1$ not including v_2 or v_3. Similarly there is no $r_1 \in V_1$, such that $v_2 \rightarrow r_1$. Setting $W_1 = (Q_1 \cup Q_2 \cup \ldots \cup Q_i) \cap V_3$ and $W_2 = (Q_{i+1} \cup Q_{i+2} \cup \ldots \cup Q_s) \cap V_3$

and using the conclusions above, we obtain $(V_1 \cup W_1) \Rightarrow (V_2 \cup W_2)$.

Suppose there exists $r_1 \in V_1 - v_1$ such that $v_3 \rightarrow r_1$. Let P be a path in $D - V(C)$ from r_1 to q_s. Then $D\langle V(P) \cup \{v_2, v_3\}\rangle$ is strong and has vertices from three partite sets. Thus, by Theorem 3.1, it has a 3-cycle. The 3-cycle does not include v_1, a contradiction. Hence, $(V_1 - v_1) \rightarrow v_3$. Similarly, we can prove that $v_3 \rightarrow (V_2 - v_2)$. This completes the proof that D satisfies the conditions in (ii). □

References

[1] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London, 2000.

[2] J. Bang-Jensen and G. Gutin, Generalizations of tournaments: a survey. J. Graph Theory **28** (1998) 171-202.

[3] J.A. Bondy, Diconnected orientation and a conjecture of Las Vergnas. J. London Math. Soc. **14** (1976) 277-282.

[4] R.J. Douglas, Tournaments that admit exactly one hamiltonian circuit. *Proc. London Math. Soc.* **21** (1970) 716-730.

[5] G.P. Egorychev, The method of coefficients and a generating function for the number of non isomorphic tournaments that have a unique hamiltonian contour, *Kombinatorniy Analis* **3** (1974) 5-9 (in Russian).

[6] M.R. Garey, On enumerating tournaments that admit exactly one hamiltonian circuit. *J. Combin. Theory Ser. B* **13** 266-269.

[7] W.D. Goddard and O.R. Oellermann, On the cycle structure of multipartite tournaments. In *Graph Theory Combin. Appl. 1* Wiley, New York (1991) 525-533.

[8] Y. Guo, Semicomplete multipartite digraphs: a generalization of tournaments. Habilitation thesis, RWTH Aachen, Germany, 1998.

[9] Y. Guo and L. Volkmann, Cycles in multipartite tournaments. J. Combin. Theory B **62** (1994) 363-366.

[10] G. Gutin, Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: a survey. J. Graph Theory **19** (1995) 481-505.

[11] G. Gutin, A note on cardinality of certain classes of unlabeled multipartite tournaments. Discrete Math. **186** (1998) 277-280.

[12] G. Gutin and A. Rafiey, Multipartite tournaments with small number of cycles. To appear in *Australasian J. Combinatorics.*

[13] J.W. Moon, The number of tournaments with a unique spanning cycle. *J. Graph Theory* **6** (1982) 303-308.

[14] L. Volkmann, Cycles in multipartite tournaments: results and problems. Discrete Math. **245** (2002) 19-53.