Polynomial algorithms for finding paths and cycles in quasi-transitive digraphs

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Abstract

A digraph D is called quasi-transitive if for any triple x, y, z of distinct vertices of D such that (x, y) and (y, z) are arcs of D there is at least one arc from x to z or from z to x. A minimum path factor of a digraph D is a collection of the minimum number of pairwise vertex disjoint paths covering the vertices of D. J. Bang-Jensen and J. Huang conjectured that there exist polynomial algorithms for the Hamiltonian path and cycle problems for quasi-transitive digraphs. We solve this conjecture by describing polynomial algorithms for finding a minimum path factor and a Hamiltonian cycle (if it exists) in a quasi-transitive digraph.

1 Introduction

A digraph D is called *quasi-transitive* if for any triple x, y, z of distinct vertices of D such that (x, y) and (y, z) are arcs of D there is at least one arc from x to z or from z to x. A digraph obtained by replacing each edge of a complete k-partite ($k \geq 2$) graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a *semicomplete k-partite digraph* or *semicomplete multipartite digraph* (abbreviated to SMD). A SMD D is called *ordinary* if, for every (ordered) pair of the partite sets X, Y such that

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there is an arc from X to Y, for each $x \in X,\ y \in Y$, (x, y) is an arc of D. A *k-path factor* of a digraph D is a collection of k pairwise vertex disjoint paths covering the vertices of D. The *path-covering number* of a digraph D ($pc(D)$) is the minimum integer k such that D has k-path factor. Obviously, D has a Hamiltonian path if and only if $pc(D) = 1$.

Quasi-transitive digraphs were introduced by Ghouilà-Houri [5] and have been studied in [1, 2, 9, 10]. Bang-Jensen and Huang [1] characterize those quasi-transitive digraphs that have a Hamiltonian cycle (Hamiltonian path, respectively) using appropriate characterizations of ordinary SMD's [6, 7]. At the same time, Bang-Jensen and Huang note that their theorems do not seem to imply polynomial algorithms and conjecture that there exist such algorithms.

In this paper, we describe $O(n^4/\log n)$-time algorithms for finding a Hamiltonian cycle (if it exists) and a $pc(D)$-path factor in a quasi-transitive digraph D on n vertices. To construct the algorithms we use a decomposition theorem that characterizes quasi-transitive digraphs in a recursive sense [1], characterizations of semicomplete multipartite digraphs containing Hamiltonian paths [6] and ordinary semicomplete multipartite digraphs having Hamiltonian cycles [7], network flow algorithms [4], and some other results.

2 Terminology and notation

The terminology is rather standard, generally following [3]. Digraphs are finite, have no loops or multiple arcs. If multiple arcs are allowed we use the term *directed multigraph.* $V(D)$ and $A(D)$ denote the vertex set and the arc set of a digraph D. A digraph D is called *transitive* if it is acyclic and for any triple x, y, z of distinct vertices of D such that (x, y) and (y, z) are arcs of D there is an arc from x to z. A digraph obtained by replacing each edge of a complete graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a *semicomplete digraph.* Obviously, a semicomplete digraph on k vertices is a semicomplete k-partite digraph. By a *cycle (path)* we mean a simple directed cycle (path, respectively). A cycle (path) of a digraph D is called *Hamiltonian* if it includes all the vertices of D. A digraph D is *strong* if there exists a path from x to y and a path from y to x in D for any choice of distinct vertices x, y of D. A collection F of pairwise vertex disjoint paths and cycles of a digraph D is called a *k-path-cycle factor* of D if F covers $V(D)$ and has exactly k paths. A 0-path-cycle factor is called a *cycle factor*. A $pc(D)$-path factor is called a *minimum* path factor.

Let D be a digraph on the n vertices $v_1, ..., v_n$ and let $L_1, ..., L_n$ be a collection of digraphs. Then $D' = D[L_1, ..., L_n]$ is the new digraph obtained from D by replacing each vertex v_i of D by L_i and by adding an arc from any vertex of L_i to any vertex of L_j if and only if (v_i, v_j) is an arc of D $(1 \leq i \neq j \leq n)$.

As usual, n will denote the number of vertices in a digraph considered.

3 Known results

Our algorithms are based on the following decomposition theorem due to Bang-Jensen and Huang [1].

Theorem 3.1 *Let D be a quasi-transitive digraph on n vertices.*

(1) If D is not strong, then there are an integer h, a transitive digraph H on h vertices, and strong quasi-transitive digraphs $S_1, ..., S_h$ such that $D = H[S_1, ..., S_h]$.

(2) If D is strong, then there exist an integer t, a semicomplete digraph T on t vertices, and non-strong quasi-transitive digraphs $Q_1, ..., Q_t$ such that $D = T[Q_1, ..., Q_t]$. Furthermore, if T has a cycle of length two induced by vertices v_i, v_j, then the corresponding digraphs Q_i and Q_j are trivial, i.e. each of them has only one vertex.

One can find the decompositions above in time $O(n^2)$.

In the next section we use also the following two theorems proved in [6, 7] (see, also, [8]).

Theorem 3.2 *Let D be a SMD.*

(1) D has a Hamiltonian path if and only if it contains a 1-path-cycle.

(2) Given a 1-path-cycle factor of D, a Hamiltonian path of D can be constructed in time $O(n^2)$.

Theorem 3.3 *Let D be a strong ordinary SMD.*

(1) D has a Hamiltonian cycle if and only if it contains a cycle factor.

(2) Suppose that D has a cycle factor. Given a cycle factor of D, a Hamiltonian cycle of D can be found in time $O(n^2)$.

4 New results

Below we consider the following more general problem instead of the Hamiltonian path one. Given a digraph D, find a minimum path factor of D. We call this problem the *MPF problem.*

Theorem 4.1 *Suppose a digraph $D = R[H_1, ..., H_r]$, $r \geq 2$, where R is either an acyclic digraph or a SMD on r vertices. Given a minimum path factor of H_i, for every $i = 1, ..., r$, the MPF problem for D can be solved in time $O(n^3/\log n)$.*

Proof: Consider the following set of digraphs

$$\mathcal{S} = \{R[E_{n_1}, ..., E_{n_r}] : \ pc(H_i) \leq n_i \leq |V(H_i)|, \ i = 1, ..., r\},$$

where E_p is a digraph of order p having no arcs. It is easy to see, that every digraph of $\mathcal{S}$ is either an acyclic digraph or a SMD. Consider, also, the network N_R containing the digraph R and two additional vertices (source and sink): s and t such that s and t are adjacent to every vertex of $V(R)$ and the arcs between s (t, resp.) and R are oriented from s to R (from R to t, resp.). Associate with a vertex v_i (corresponding to H_i) of R the lower and upper bounds $pc(H_i)$ and $|V(H_i)|$ ($i = 1, ..., r$).

Suppose that N_R admits a flow f of value $k \geq 1$. Then there is a collection L_k of k paths and a number of cycles covering $V(R)$. Indeed, construct a directed multigraph M on the vertices $v_1, ..., v_n, s, t$ as follows. The number of arcs from a vertex u of M to another one w is equal to the number of units of f in the arc (u, w) of N_R. Merging vertices s and t in M, we obtain an Eulerian directed multigraph M^*. Since M^* contains an Euler tour, M has the collection L_k above.

Since a vertex v_i of R lies on t_i of paths and cycles of L_k, for some t_i such that $pc(H_i) \leq t_i \leq |V(H_i)|$, we can transform L_k into a k-path-cycle factor $F(L_k)$ of a digraph $Q = R[E_{t_1}, \ldots, E_{t_r}] \in \mathcal{S}$ by replacing the vertex v_i by t_i

independent new vertices such that each new vertex corresponds to one of the occurrences of v_i in L_k. Since $Q \in \mathcal{S}$, one can transform, in polynomial time, $F(L_k)$ into a k-path factor $F'(L_k)$ of Q. Indeed, if Q is acyclic this is trivial. If Q is semicomplete multipartite, then this follows from Theorem 3.2: replace a path and all the cycles of $F(L_k)$ by a path. Finally change $F'(L_k)$ to a k-path factor $F''(L_k)$ of D, by replacing the vertices of each E_{t_i} by t_i paths that form a t_i-path factor of H_i.

Conversely, suppose P_k is a k-path factor of D. For each H_i, $A(H_i) \cap A(P_k)$ induce a collection of α_i vertex disjoint paths in H_i. Clearly $pc(H_i) \leq \alpha_i \leq |V(H_i)|$. Let $Q = R[E_{\alpha_1}, \ldots, E_{\alpha_r}] \in \mathcal{S}$. Then $Q(P_k)$ has a k-path factor which can be obtained from P_k by contracting, for all i, each of the α_i subpaths in H_i to a vertex. It is easy to check that if a digraph from $\mathcal{S}$ has k-path factor, then N_R admits a flow of value k.

Hence, $pc(D) = max\{1, m\}$, where m is the value of a minimum flow in N_R. Now, given $pc(H_1), ..., pc(H_r)$ and corresponding path factors, the MPF problem for D can be solved as follows. Construct N_R and the following feasible flow g of it. For every $i = 1, ..., r$, $g(sv_i) = g(v_i t) = pc(H_i)$ and, for every pair i, j $(1 \leq i \neq j \leq r)$, $g(v_i v_j) = 0$. Find a minimum flow f from s to t (= a maximum flow from t to s). It is clear that f can be found in time $O(n^3/\log n)$ [4]. Using f, a minimum path factor $F''(L_{pc(D)})$ of D can be constructed as in the proof above.

□

Theorem 4.2 *The MPF problem for a quasi-transitive digraph D can be solved in time $O(n^4/\log n)$.*

Proof: To prove this theorem we just give the following recursive algorithm APF for solving the MPF problem for a quasi-transitive digraph D.

1. Find a decomposition $D = R[H_1, ..., H_r]$, $r \geq 2$ (see Theorem 3.1), where R is either transitive or semicomplete.

2. For every $i = 1, ..., r$, if $|V(H_i)| = 1$, then take H_i as a minimum path factor of itself, otherwise call APF to construct a minimum path factor of H_i.

3. Using the algorithm described in Theorem 4.1 find a minimum path factor of D.

It is easy to see that the complexity of the algorithm above is $O(n^4/\log n)$.
□

Theorem 4.3 *The Hamiltonian cycle problems for a quasi-transitive digraph D can be solved in time $O(n^4/\log n)$.*

Proof: To prove this theorem we give the following algorithm for solving the Hamiltonian cycle problem for a strong quasi-transitive digraph D.

1. Find a decomposition $D = R[H_1, ..., H_r]$ (see Theorem 3.1), where R is either transitive or semicomplete.

2. For every $i = 1, ..., r$, find a minimum path factor of H_i by the algorithm from Theorem 4.2.

3. Find a minimum flow f in the network N_R (see the proof of Theorem 4.1). If the value of f is not 0, then D has no Hamiltonian cycle. Otherwise, using f construct a cycle factor F of some $Q \in \mathcal{S}$ (see the proof of Theorem 4.1). Transform F into a Hamiltonian cycle H of Q using the algorithm from Theorem 3.3 (Q is an ordinary SMD). Transform H into a Hamiltonian cycle of D.

It is not difficult to check that the complexity of the algorithm above is $O(n^4/\log n)$. □

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References

[1] J. Bang-Jensen and J. Huang, Quasi-transitive digraphs, Preprint, no. 27, Inst. Mat. Datalogi, Odense Univ., 1993.

[2] J. Bang-Jensen and J. Huang, Quasi-kings in quasi-transitive digraphs, Preprint, no. 33, Inst. Mat. Datalogi, Odense Univ., 1993.

[3] J. A. Bondy and U.R.S. Murty *Graph Theory with Applications* (North Holland, N. Y., 1976).

[4] J. Cheriyan, T. Hagerup, and K. Mehlhorn, Can a maximum flow be computed in $o(nm)$ time? in Proc. of 17th ICALP, 1990, 235-248.

[5] A. Ghouilà-Houri, Caractérisation des graphes non orientès dont on peut orienter les arrêtes de maniere à obtenir le graphe d'un relation d'ordre, *C.R.Acad. Sci.Paris* **254** (1962) 1370-1371.

[6] G. Gutin, Finding a longest path in a complete multipartite digraph, *SIAM J. Discrete Math.*, **6** (1993) 270-273.

[7] G. Gutin *Paths and cycles in digraphs.* Ph.D thesis, Tel Aviv Univ., 1993.

[8] G. Gutin, Cycles and paths in complete multipartite digraphs, theorems and algorithms: a survey, *J. Graph Theory*, to appear.

[9] P. Hell and J. Huang, Lexicographic orientation and representation algorithms for comparability graphs, proper circular graphs, and proper interval graphs, submitted.

[10] J. Huang, Tournament-like oriented graphs, *Ph.D. thesis*, Simon Fraser Univ., 1992.