# Minimum Cost Homomorphism Dichotomy for Oriented Cycles

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**Abstract.** For digraphs $D$ and $H$, a mapping $f: V(D) \rightarrow V(H)$ is a homomorphism of $D$ to $H$ if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. If, moreover, each vertex $u \in V(D)$ is associated with costs $c_i(u), i \in V(H)$, then the cost of the homomorphism $f$ is $\sum_{u \in V(D)} c_{f(u)}(u)$. For each fixed digraph $H$, we have the *minimum cost homomorphism problem for $H$* (abbreviated MinHOM($H$)). In this discrete optimization problem, we are to decide, for an input graph $D$ with costs $c_i(u)$, $u \in V(D), i \in V(H)$, whether there exists a homomorphism of $D$ to $H$ and, if one exists, to find one of minimum cost. We obtain a dichotomy classification for the time complexity of MinHOM($H$) when $H$ is an oriented cycle. We conjecture a dichotomy classification for all digraphs with possible loops.

## 1 Introduction

For directed (undirected) graphs $G$ and $H$, a mapping $f: V(G) \rightarrow V(H)$ is a *homomorphism of $G$ to $H$* if $uv$ is an arc (edge) implies that $f(u)f(v)$ is an arc (edge). Let $H$ be a fixed directed or undirected graph. The *homomorphism problem* for $H$ asks whether a directed or undirected input graph $G$ admits a homomorphism to $H$. The *list homomorphism problem* for $H$ asks whether a directed or undirected input graph $G$ with lists (sets) $L_u \subseteq V(H), u \in V(G)$ admits a homomorphism $f$ to $H$ in which $f(u) \in L_u$ for each $u \in V(G)$.

Suppose $G$ and $H$ are directed (or undirected) graphs, and $c_i(u)$, $u \in V(G)$, $i \in V(H)$ are nonnegative *costs*. The *cost of a homomorphism* $f$ of $G$ to $H$ is $\sum_{u \in V(G)} c_{f(u)}(u)$. If $H$ is fixed, the *minimum cost homomorphism problem*, MinHOM($H$), for $H$ is the following discrete optimization problem. Given an input graph $G$, together with costs $c_i(u)$, $u \in V(G)$, $i \in V(H)$, we wish to find a minimum cost homomorphism of $G$ to $H$, or state that none exists.

The minimum cost homomorphism problem was introduced in [10], where it was motivated by a real-world problem in defence logistics. We believe it offers a practical and natural model for optimization of weighted homomorphisms. The problem's special cases include the homomorphism and list homomorphism problems [15, 17] and the general optimum cost chromatic partition problem, which has been intensively studied [13, 19, 20].

There is an extensive literature on the minimum cost homomorphism problem, e.g., see [5–10]. These and other papers study the time complexity of MinHOM($H$) for various families of directed and undirected graphs. In particular, Gutin, Hell, Rafiey and Yeo [6] proved a dichotomy classification for all undirected graphs (with possible loops): If $H$ is a reflexive proper interval graph or a proper interval bigraph, then MinHOM($H$) is polynomial time solvable; otherwise, MinHOM($H$) is NP-hard. It is an open problem whether there is a dichotomy classification for the complexity of MinHOM($H$) when $H$ is a digraph with possible loops. We conjecture that such a classification exists and, moreover, the following assertion holds:

*Conjecture 1.* Let $H$ be a digraph with possible loops. Then MinHOM($H$) is polynomial time solvable if $H$ has either a Min-Max ordering or a $k$-Min-Max ordering for some $k \geq 2$. Otherwise, MinHOM($H$) is NP-hard.

For the definitions of a Min-Max and $k$-Min-Max ordering see Section 3, where we give theorems (first proved in [10, 9]) showing that if $H$ has one of the two orderings, then MinHOM($H$) is polynomial time solvable. So, it is the NP-hardness part of Conjecture 1 which is the 'open' part of the conjecture.

Very recently Gupta, Hell, Karimi and Rafiey [5] obtained a dichotomy classification for all reflexive digraphs that confirms this conjecture. They proved that if a reflexive digraph $H$ has no Min-Max ordering, then MinHOM($H$) is NP-hard. Gutin, Rafiey and Yeo [8, 9] proved that if a semicomplete multipartite digraph $H$ has neither Min-Max ordering nor $k$-Min-Max ordering, then MinHOM($H$) is NP-hard.

In this paper, we show that the same result (as for semicomplete multipartite digraphs) holds for oriented cycles. This provides a further support for Conjecture 1. In fact, we prove a graph-theoretical dichotomy for the complexity of MinHOM($H$) when $H$ is an oriented cycle. The fact that Conjecture 1 holds for oriented cycles follows from the proof of the graph-theoretical dichotomy. In the proof, we use a new concept of a $(k, l)$-Min-Max ordering introduced in Section 3. Our motivation for Conjecture 1 partially stems from the fact that we initially proved polynomial time solvability of MinHOM($H$) when $V(H)$ has a $(k, l)$-Min-Max ordering by reducing it to the minimum cut problem. However, we later proved that $(k, l)$-Min-Max orderings can simply be reduced to $p$-Min-Max orderings for $p \geq 1$ (see Section 3).

Homomorphisms to oriented cycles have been investigated in a number of papers. Partial results for the homomorphism problem to oriented cycles were obtained in [11] and [18]. A full dichotomy was proved by Feder [3]. Feder, Hell and Rafiey [4] obtained a dichotomy for the list homomorphism problem for oriented cycles. Notice that our dichotomy is different from the ones in [3] and [4].

Bulatov [2] proved that there exists a dichotomy classification for the list homomorphism problem for digraphs, but no such dichotomy has been obtained and even conjectured to the best of our knowledge. For the homomorphism problem for digraphs, we do not even know whether a dichotomy exists and there is no conjecture of such a classification for the general case.

The rest of this paper is organized as follows. In the next section we consider so-called levels of vertices in oriented paths and cycles. The concepts of Min-Max ordering, $k$-Min-Max ordering and $(k, l)$-Min-Max ordering are considered in Section 3. In Section 4 we obtain a dichotomy classification for MinHOM($H$) when $H$ is a balanced oriented cycle. For all oriented cycles $H$, a dichotomy is proved in Section 5.

## 2 Levels of Vertices in Oriented Paths and Cycles

In this paper $[p]$ denotes the set $\{1, 2, \ldots, p\}$. Let $D$ be a digraph. We will use $V(D)$ ($A(D)$) to denote the vertex (arc) set of $D$. We say that $xy$ ($x, y \in V(D)$) is an *edge* of $D$ if either $xy$ or $yx$ is an arc of $D$. A sequence $b_1b_2 \ldots b_p$ of distinct vertices of $D$ is an *oriented path* if $b_ib_{i+1}$ is an edge for every $i \in [p-1]$. If $b_1b_2 \ldots b_p$ is an oriented path, we call $C = b_1b_2 \ldots b_pb_1$ an *oriented cycle* if $b_pb_1$ is an edge. An edge $b_ib_{i+1}$ (here $b_pb_{p+1} = b_pb_1$) of an oriented path $P$ or cycle $C$ is called *forward* (*backward*) if $b_ib_{i+1} \in A(D)$ ($b_{i+1}b_i \in A(D)$).

Let $P = b_1b_2 \ldots b_p$ be an oriented path. We assign *levels* to the vertices of $P$ as follows: we set $\mathrm{level}_P(b_1) = 0$, and $\mathrm{level}_P(b_{t+1}) = \mathrm{level}_P(b_t) + 1$, if $b_tb_{t+1}$ is forward and and $\mathrm{level}_P(b_{t+1}) = \mathrm{level}_P(b_t) - 1$, if $b_tb_{t+1}$ is backward. We say that $P$ is *of type* $r$ if $r = \max\{\mathrm{level}_P(b_i) :\ i \in [p]\} = \mathrm{level}_P(b_p)$ and $0 \le \mathrm{level}_P(b_t) \le r$ for each $t \in [p]$.

An oriented cycle $C$ is *balanced* if the number of forward edges equals the number of backward edges; if $C$ is not balanced, it is called *unbalanced*. Note that the fact whether $C$ is balanced or unbalanced does not depend on the choice of the vertex $b_1$ or the direction of $C$.

Let $C = b_1b_2 \ldots b_pb_1$ be an oriented cycle. It has two *directions*: $b_1b_2 \ldots b_pb_1$ and $b_1b_pb_{p-1} \ldots b_1$. In what follows, we will always consider the direction in which the number of forward arcs is no smaller than the number of backward arcs. We can assign *levels* to the vertices of $C$ as follows: $\mathrm{level}(b_1) = k$, where $k$ is a non-negative integer, and $\mathrm{level}(b_{t+1}) = \mathrm{level}(b_t) + 1$, if $b_tb_{t+1}$ is forward and and $\mathrm{level}(b_{t+1}) = \mathrm{level}(b_t) - 1$, if $b_tb_{t+1}$ is backward. Clearly, the value of each $\mathrm{level}(b_i)$, $i \in [p]$, depends on both $k$ and the choose of the initial vertex $b_1$. Feder [3] proved the following useful result.

**Proposition 1.** *The integer $k$ and initial vertex $b_1$ in an oriented cycle $C$ can be chosen such that* $\text{level}(b_1) = 0$ *and* $\text{level}(b_i) \geq 0$ *for every* $i \in [p]$*. If $C$ is unbalanced, then $k$ and $b_1$ can be chosen such that* $\text{level}(b_1) = 0$ *and* $\text{level}(b_i) > 0$ *for every* $i \in [p] \setminus \{1\}$.

Since the proposition was proved in [3], we will not give its complete proof. Instead, we will outline a procedure for finding appropriate $k$ and $b_1$ and remark on how the procedure can be used in showing the proposition.

Let $C = b_1b_2 \ldots b_pb_1$ be an oriented cycle. We may assume that $b_1$ is chosen in such a way that if $C$ has a backward edge, then $b_pb_1$ is a backward edge. Compute $m_i$, the number of the forward arcs minus the number of backward arcs in the oriented path $b_1b_2 \ldots b_i$, for each $i \in [p]$. Set $k = |\min\{m_i : i \in [p]\}|$. Assign the level to each vertex of $C$ using the level definition and starting from assigning level $k$ to $b_1$. By the definition of $k$, the level of each vertex $b_j$ is non-negative and there are vertices $b_i$ of level zero. Choose such a vertex $b_i$ with maximum index $i$ and reassign the levels to the vertices of $C$ as follows. Consider $C' = b_ib_{i+1} \ldots b_pb_1b_2 \ldots b_i$ and set $\text{level}(b_i) = 0$ and the rest of the levels according to the order of vertices given in $C'$.

This procedure can be turned into a proof of the proposition by observing that if $C'$ is unbalanced, then the level of $b_1$ in $C'$ will be greater than the level of $b_1$ in $C$. Thus, the levels of all vertices $v_j$, $j \in [i-1]$ will be greater than their levels in $C$, implying that the only level zero vertex in $C'$ is $b_i$.

Thus, in the rest of the paper, we may assume that the 'first' vertex of $b_1$ of an oriented cycle $C = b_1b_2 \ldots b_pb_1$ is chosen in such a way that the levels of all vertices of $C$ satisfy Proposition 1.

We will extensively use the following notation: $VL(C) = \{b_t : \text{level}(b_t) = 0, t \in [p]\}$, $h(C) = \max\{\text{level}(b_j) : j \in [p]\}$, and $VH(C) = \{b_t : \text{level}(b_t) = h(C), t \in [p]\}$. Note that for unbalanced cycles $C$ we have $|VL(C)| = 1$.

The concepts of this section are illustrated on Figure 1. In particular, $Z$ is balanced with forward edges $b_1b_2, b_3b_4, \ldots$ and backward edges $b_2b_3, b_4b_5, \ldots$. We have $\text{level}(b_1) = \text{level}(b_3) = \text{level}(b_5) = 0$, $\text{level}(b_2) = \text{level}(b_4) = \text{level}(b_6) = \text{level}(b_8) = \text{level}(b_{14}) = 1$, $\text{level}(b_7) = \text{level}(b_9) = \text{level}(b_{11}) = \text{level}(b_{13}) = 2$ and $\text{level}(b_{10}) = \text{level}(b_{12}) = 3$. Thus, $h(Z) = 3$, $VL(Z) = \{b_1, b_3, b_5\}$, $VH(Z) = \{b_{10}, b_{12}\}$.

## 3 $k$-Min-Max and $(k, l)$-Min-Max Orderings

All known polynomial cases of MinHOM($H$) can be formulated in terms of certain vertex orderings. In fact, all known polynomial cases can be partitioned into two classes: digraphs $H$ admitting a Min-Max ordering of their vertices and digraphs $H$ having a $k$-Min-Max ordering of their vertices ($k \geq 2$). Both types of orderings are defined in this section, where we also introduce a new

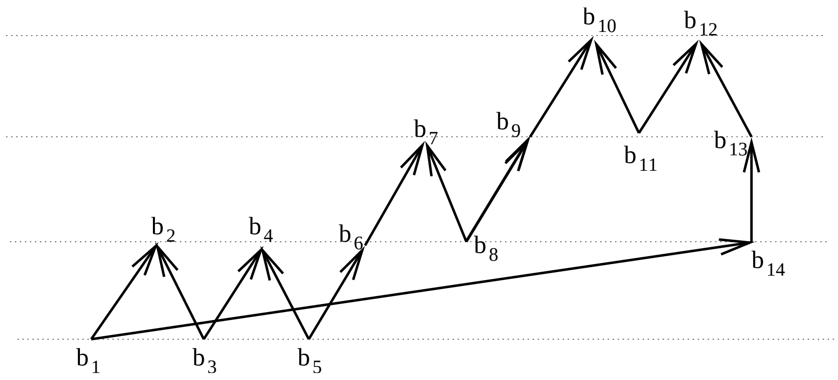


**Fig. 1.** A level diagram of oriented cycle $Z = b_1b_2 \dots b_{14}b_1$, where $b_1b_2, b_3b_2, b_3b_4, b_5b_4, b_5b_6, b_6b_7, b_8b_7b_8b_9, b_9b_{10}, b_{11}b_{10}, b_{11}b_{12}, b_{13}b_{12}, b_{14}b_{13}, b_1b_{14}$ are arcs.

type of ordering, a $(k, l)$-Min-Max ordering. It may be surprising, but we prove that the new type of ordering can be reduced to the two known orderings.

Let $H$ be a digraph and let $(v_1, v_2, \dots, v_p)$ be an ordering of the vertices of $H$. Let $e = v_iv_r$ and $f = v_jv_s$ be two arcs in $H$. The pair $v_{\min\{i,j\}}v_{\min\{s,r\}}$ ($v_{\max\{i,j\}}v_{\max\{s,r\}}$) is called the *minimum* (*maximum*) of the pair $e, f$. (The minimum (maximum) of two arcs is not necessarily an arc.) An ordering $(v_1, v_2, \dots, v_p)$ is a *Min-Max ordering of* $V(H)$ if both minimum and maximum of every two arcs in $H$ are in $A(H)$. Two arcs $e, f \in A(H)$ are called a *crossing pair* if $\{e, f\} \neq \{g', g''\}$, where $g'$ ($g''$) is the minimum (maximum) of $e, f$. Clearly, to check that an ordering is Min-Max, it suffices to verify that the minimum and maximum of every crossing pair of arcs are arcs, too. The concept of Min-Max ordering is of interest due to the following:

**Theorem 1.** *[10] If a digraph $H$ has a Min-Max ordering of $V(H)$, then MinHOM($H$) is polynomial-time solvable.*

We will sometimes call a Min-Max ordering also a *1-Min-Max ordering.* The reason for this will become apparent in the rest of this section.

A collection $V_1, V_2, \dots V_k$ of subsets of a set $V$ is called a *$k$-partition* of $V$ if $V = V_1 \cup V_2 \cup \dots \cup V_k$, $V_i \cap V_j = \emptyset$ provided $i \neq j$.

Let $H = (V, A)$ be a digraph and let $k \geq 2$ be an integer. We say that $H$ has a *$k$-Min-Max ordering* of $V(H)$ if there is a $k$-partition of $V$ into subsets $V_1, V_2, \dots V_k$ and there is an ordering $\boldsymbol{V}_i = (v^i_1, v^i_2, \dots, v^i_{\ell(i)})$ of $V_i$ for each $i$ such that

**(i)** Every arc of $H$ is an arc from $V_i$ to $V_{i+1}$ for some $i \in [k]$ and
**(ii)** $(\boldsymbol{V}_i, \boldsymbol{V}_{i+1}) = (v^i_1, v^i_2, \dots, v^i_{\ell(i)}v^{i+1}_1v^{i+1}_2, \dots, v^{i+1}_{\ell(i+1)})$ is a Min-Max ordering of the subdigraph of $H$ induced by $V_i \cup V_{i+1}$ for each $i \in [k]$. (All indices are taken modulo $k$.)

In such a case, $(\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_k)$ is a *$k$-Min-Max ordering* of $V(H)$; $k$-Min-Max orderings are of interest due to the following:

**Theorem 2.** *[9] If a digraph $H$ has a $k$-Min-Max ordering of $V(H)$, then MinHOM($H$) is polynomial-time solvable.*

Our study of MinHOM($H$) for oriented cycles $H$ has led us to the following new concept.

**Definition 1.** *Let $H = (V, A)$ be a digraph and let $k \geq 2$ and $l$ be integers. For $l < k$ we say that $H$ has a* $(k, l)$-Min-Max ordering *if there is a $(k+l-2)$-partition of $V$ into subsets $V_1, V_2, \ldots, V_k, U_2, U_3, \ldots, U_{l-1}$ (set $U_1 = V_1$, $U_l = V_k$) and there is an ordering $\boldsymbol{V}_i = (v^i_1, v^i_2, \ldots, v^i_{\ell^v(i)})$ of $V_i$ for each $1 \leq i \leq k$ and there is an ordering $\boldsymbol{U}_i = (u^i_1, u^i_2, \ldots, u^i_{\ell^u(i)})$ of $U_i$ for each $1 \leq i \leq l$ such that*

- **(i)** *Every arc of $H$ is an arc from $V_i$ to $V_{i+1}$) for some $i \in [k-1]$, or is an arc from $U_j$ to $U_{j+1}$ for some $j \in [l-1]$.*
- **(ii)** *$(\boldsymbol{V}_i, \boldsymbol{V}_{i+1})$ is a Min-Max ordering of the subdigraph $H[V_i \cup V_{i+1}]$ for all $i \in [k-1]$.*
- **(iii)** *$(\boldsymbol{U}_i, \boldsymbol{U}_{i+1})$ is a Min-Max ordering of the subdigraph $H[U_i \cup U_{i+1}]$ for all $i \in [l-1]$.*
- **(iv)** *$(\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{U}_2)$ is a Min-Max ordering of the subdigraph $H[V_1 \cup V_2 \cup U_2]$.*
- **(v)** *$(\boldsymbol{U}_{l-1}, \boldsymbol{V}_{k-1}, \boldsymbol{V}_k)$ is a Min-Max ordering of the subdigraph $H[V_{k-1} \cup U_{l-1} \cup V_k]$.*

It turns out that $(k, l)$-Min-Max orderings can be reduced to $p$-Min-Max orderings as follows from the next assertion:

**Theorem 3.** *If a digraph $H$ has a $(k, l)$-Min-Max ordering, then MinHOM($H$) is polynomial-time solvable.*

*Proof.* Let $H$ have a $(k, l)$-Min-Max ordering as described in Definition 1. Let $d = k - l$. We will show that $H$ has a $d$-Min-Max ordering, which will be sufficient because of Theorems 1 and 2. (Recall that a 1-Min-Max ordering is simply a Min-Max ordering.) Let us consider two cases.

**Case 1:** $d = 1$. It is not difficult to show that the ordering

$$(\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{U}_2, \boldsymbol{V}_3, \boldsymbol{U}_3, \ldots, \boldsymbol{V}_{k-2}, \boldsymbol{U}_{k-2}, \boldsymbol{V}_{k-1}, \boldsymbol{V}_k)$$

is a Min-Max ordering. Indeed, all crossing pairs of arcs are only in the subgraphs given in (ii)-(v) of Definition 1. According to the definition, the maximum and minimum of every crossing pair is in $H$.

**Case 2:** $d \geq 2$. Let $s_i = \max\{p : \ i + pd \leq l\}$ for each $i \in [d]$. Consider the following orderings for each $2 \leq i \leq d$

$$\boldsymbol{W_i} = (\boldsymbol{V}_i, \boldsymbol{U}_i, \boldsymbol{V}_{i+d}, \boldsymbol{U}_{i+d}, \boldsymbol{V}_{i+2d}, \boldsymbol{U}_{i+2d}, \ldots, \boldsymbol{V}_{i+s_i d}, \boldsymbol{U}_{i+s_i d}, \boldsymbol{V}_{i+(s_i+1)d})$$

and the ordering

$$\boldsymbol{W_1} = (\boldsymbol{V}_1, \boldsymbol{V}_{1+d}, \boldsymbol{U}_{1+d}, \boldsymbol{V}_{1+2d}, \boldsymbol{U}_{1+2d}, \ldots, \boldsymbol{V}_{1+s_1 d}, \boldsymbol{U}_{1+s_1 d}, \boldsymbol{V}_{1+(s_1+1)d}).$$

Observe that $W_1, W_2, \ldots, W_d$ form a partition of $V(H)$ and that every arc is from $W_i$ to $W_{i+1}$ for some $i \in [d]$, where $W_{d+1} = W_1$. As in Case 1, it is not difficult to see that $(\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_d)$ is a $d$-Min-Max ordering of $H$. □

*Remark 1.* Notice that not always a $p$-Min-Max ordering can be reduced to a $(k,l)$-Min-Max ordering. As an example, consider the directed cycle $\boldsymbol{C}_5$.

## 4 Balanced Oriented Cycles

We say that a balanced oriented cycle $C = b_1 b_2 \ldots b_p b_1$ is *of the form* $(l^+h^+)^q$ with $q \geq 1$ if $P = C - b_p b_1$ can be written as $P = P_1 R_1 P_2 R_2 \ldots P_q R_q$, where

$$V(P_i) \cap VL(C) \neq \emptyset,\ V(P_i) \cap VH(C) = \emptyset,$$
$$V(R_i) \cap VL(C) = \emptyset,\ V(R_i) \cap VH(C) \neq \emptyset$$

for each $i \in [q]$. We write $l^+h^+$ instead of $(l^+h^+)^1$. For example, the cycle in Figure 1 is of the form $l^+h^+$. Balanced oriented cycles $C$ of the form $l^+h^+$ are considered in the following:

**Theorem 4.** *Let $C = b_1 b_2 \ldots b_p b_1$ be a balanced oriented cycle of the form $l^+h^+$. Then MinHOM($C$) is polynomial time solvable.*

*Proof.* Let $q = \min\{j : \ b_j \in VH(C)\}$, let $m = h(C)$ and let $V = V(C)$. Consider the following ordering $\boldsymbol{V} = (b_p, b_{p-1}, \ldots, b_{q+1}, b_1, b_2 \ldots, b_q)$ of $V$. We can define the following natural $(m+1)$-partition of $V$: $V_1, V_2, \ldots, V_{m+1}$, where $V_j = \{b_s \in V(C) : \ \mathrm{level}(b_s) = j-1\}$. Note that every arc of $C$ is an arc from $V_j$ to $V_{j+1}$ for some $j \in [m]$.

Let $\boldsymbol{V}_j$ be the ordering of $V_j$ obtained from $\boldsymbol{V}$ by deleting all vertices not in $V_j$ and let $\boldsymbol{V}_j = (s_1^j, s_2^j, \ldots, s_{b(j)}^j)$. Observe that the digraph $C[V_j \cup V_{j+1}]$ has no crossing pair of arcs for any $j \in [m]$ since, for every pair $s_\alpha^j s_\beta^{j+1}$, $s_\gamma^j s_\delta^{j+1}$ of arcs in the digraph, we have that either $\alpha \leq \gamma$ and $\beta \leq \delta$, or $\alpha \geq \gamma$ and $\beta \geq \delta$. Thus, $C$ has an $(m+1)$-Min-Max ordering of vertices and, by Theorem 2, MinHOM($C$) is polynomial-time solvable. □

The following lemma was first proved in [12]; see also [3, 21] and Lemma 2.36 in [17].

**Lemma 1.** *Let $P_1$ and $P_2$ be two oriented paths of type $r$. Then there is an oriented path $P$ of type $r$ that maps homomorphically to $P_1$ and $P_2$ such that the initial vertex of $P$ maps to the initial vertices of $P_1$ and $P_2$ and the terminal vertex of $P$ maps to the terminal vertices of $P_1$ and $P_2$. The length of $P$ is polynomial in the lengths of $P_1$ and $P_2$.*

We need a modified version of Lemma 1, Lemma 2. We say that an oriented path $b_1b_2\dots b_p$ of type $r$ is *of the form* $(l^+h^+)^k$ if the balanced oriented cycle

$$b_1b_2\dots b_pa_{r-1}a_{r-2}\dots a_2a_1b_1$$

is of the form $(l^+h^+)^k$, where $b_1a_1a_2\dots a_{r-2}a_{r-1}b_p$ is a directed path.

**Lemma 2.** *Let $P_1$ and $P_2$ be two oriented paths of type $r$. Let $P_1$ be of the form $h^+l^+$ and let $P_2$ be of the form $(l^+h^+)^k$, $k \geq 1$. Then there is an oriented path $P$ of type $r$ that maps homomorphically to $P_1$ and $P_2$ such that the initial vertex of $P$ maps to the initial vertices of $P_1$ and $P_2$ and the terminal vertex of $P$ maps to the terminal vertices of $P_1$ and $P_2$. The length of $P$ is polynomial in the lengths of $P_1$ and $P_2$, and $P$ is of the form $(l^+h^+)^k$.*

*Proof.* We will show that our construction implies that $|V(P)| \leq |V(P_1)| \times |V(P_2)|$.

We first prove the lemma for the case when $k = 1$. The proof is by induction on $r \geq 0$. If $0 \leq r \leq 1$, the claim is trivial. Assume that $r \geq 2$. Let $P_1 = a_1a_2\dots a_p$, let $P_2 = b_1b_2\dots b_q$, let $s_1 = \min\{i : \text{level}_{P_1}(a_i) = r\}$ and let $s_2 = \min\{i : \text{level}_{P_2}(b_i) = r\}$. Let $\beta_1 = \min\{\text{level}_{P_1}(a_i) : s_1 \leq i\}$ and $\beta_2 = \min\{\text{level}_{P_2}(b_i) : s_2 \leq i\}$. Without loss of generality assume that $\beta_1 \leq \beta_2$ and let $t_1 = \min\{i : \text{level}_{P_1}(a_i) = \beta_1 \text{ and } i \geq s_1\}$ and let $s_2 = \max\{i : \text{level}_{P_2}(b_i) = \beta_1 \text{ and } i \leq s_2\}$. Note that $\beta_1 > 1$ as $P_1$ is of form $h^+l^+$.

By the induction hypothesis, there is an appropriate oriented path $P'$ that can be mapped homomorphically to $a_1a_2\dots a_{s_1-1}$ and $b_1b_2\dots b_{s_2-1}$. There is also an oriented path $P''$ that can be mapped homomorphically to $a_{s_1}a_{s_1+1}\dots a_{t_1}$ and $b_{s_2}b_{s_2-1}\dots b_{t_2}$ (by reversing the two paths and then reversing the path we get by the induction hypothesis). Furthermore there is an oriented path $P'''$ that can be mapped homomorphically to $a_{t_1+1}a_{t_1+2}\dots a_p$ and $b_{t_2+1}b_{t_2+2}\dots b_p$. Let $P = P'P''P'''$ (where the arc between the last vertex of $P'$ to the first vertex of $P''$ is oriented from $P'$ to $P''$ and similarly the arc between $P''$ and $P'''$ is oriented in that direction). Note that $P$ is of type $r$ and form $h^+l^+$ and maps homomorphically to $P_1$ and $P_2$ such that the initial vertex of $P$ maps to the initial vertices of $P_1$ and $P_2$ and the terminal vertex of $P$ maps to the terminal vertices of $P_1$ and $P_2$. Furthermore $|V(P)| \leq (s_1-1)(s_2-1) + (t_1 - s_1+1)(s_2-t_2+1)+(p-s_1)(q-t_2)$. As $(s_1-1)+(t_1-s_1+1)+(p-s_1) = p$ and $(s_2-1), (s_2-t_2+1), (q-t_2) \leq q$ we have $|V(P)| \leq pq \leq |V(P_1)| \times |V(P_2)|$.

Now we proceed by induction on $k \geq 1$. The base case has already been proved. Assume that $k \geq 2$ and let $P_1 = a_1a_2 \ldots a_p$ and $P_2 = b_1b_2 \ldots b_q$. Let $t = \max\{i : \text{level}_{P_2}(b_i) = 0\}$ and let $s = \max\{i : i < t, \text{level}_{P_2}(b_i) = r\}$. By the induction hypothesis, there is an appropriate oriented path $P'$ that can be mapped homomorphically to $P_1$ and $b_1b_2 \ldots b_s$. Also, there is an appropriate oriented path $P''$ ($P'''$) that can be mapped homomorphically to $a_pa_{p-1} \ldots a_1$ and $b_sb_{s+1} \ldots b_t$ ($P_1$ and $b_tb_{t+1} \ldots b_q$). Now obtain a new oriented path $P$ by identifying the terminal vertex of $P'$ with the initial vertex of $P''$ and the terminal vertex of $P''$ with the initial vertex of $P'''$. Observe that $P$ satisfies the required properties. □

Consider the oriented cycle $C_4^0 = 12341$ with arcs $12, 32, 14, 34$. Observe that $1, 2, 3, 4$ is a Min-Max ordering of $V(C_4^0)$ and, thus, MinHOM($C_4^0$) is polynomial-time solvable.

**Theorem 5.** *Let $C = b_1b_2 \ldots b_pb_1$ be a balanced oriented cycle of the form $(l^+h^+)^k$ , $k \geq 2$, and let $C \neq C_4^0$. Then MinHOM($C$) is NP-hard.*

*Proof.* Let $C \neq C_4^0$. Let $s = \min\{j : b_j \in VH(C)\}$, $q = \min\{j : j > s, b_j \in VL(C)\}$, $t = \min\{j : j > q, b_j \in VH(C)\}$ and $m = h(C)$. Let $P_1 = b_1b_2 \ldots b_s$, $P_2 = b_qb_{q-1} \ldots b_s$, $P_3 = b_qb_{q+1} \ldots b_t$ and $P_4 = b_1b_pb_{p-1} \ldots b_t$. Note that each $P_j$ is of type $m$. By Lemma 2 there is a path $Q_1$ of type $m$ which is mapped homomorphically to $P_4$, $P_2$ and $P_3$. There is also a path $Q_2$ of type $m$ and which is mapped homomorphically to $P_1$ and $P_3$. Since $C \neq C_4^0$ and the end-vertices vertices of $Q_1$ are mapped to the end-vertices vertices of $P_4$, the path $Q_1$ contains more than two vertices. Furthermore, by Lemma 2 we may assume that $Q_1$ is of the form $(l^+h^+)^{k-1}$ and $Q_2$ is of the form $l^+h^+$.

Let $x$ ($y$) be the terminal vertex of $Q_1$ ($Q_2$). Form a new oriented path $Q = q_1q_2 \ldots q_l$ by identifying $x$ with $y$ and let $1 \leq r \leq l$ be defined such that $Q_1 = q_1q_2 \ldots q_r$ and $Q_2 = q_lq_{l-1} \ldots q_r$. As $Q_1$ contains more than two vertices we have that $r \geq 3$.

Let $D$ be an arbitrary digraph. We will now reduce the problem of finding a maximum independent set in $D$ (i.e. in the underlying graph of $D$) to MinHOM($C$). Replace every arc $ab$ of $D$ by a copy of $Q$ identifying $q_1$ with $a$ and $q_l$ with $b$, and denote the obtained digraph by $D'$. For every path $Q$ in $D'$ which we added in the construction of $D'$ we define the cost function $c$ as follows, where $M$ is a number greater than $|V(D)|$:

**(i)** $c_{b_1}(q_1) = c_{b_1}(q_l) = 0$ and $c_{b_q}(q_1) = c_{b_q}(q_l) = 1$;
**(ii)** $c_{b_s}(q_r) = c_{b_t}(q_r) = 0$ and $c_b(q_r) = M$ for all $b \in V(C) - \{b_s, b_t\}$;
**(iii)** $c_{b_2}(q) = M$ for all $q \in \{q_2, q_3, \ldots, q_{r-1}\}$ ($\neq \emptyset$, as $r \geq 3$);
**(iv)** if $s = 2$ then $c_{b_1}(q_{r-1}) = M$;
**(v)** all other costs of mapping vertices of $D'$ to $H$ are zero.

Consider a mapping $g$ from $V(Q)$ to $V(C)$, where $g(q_1) = b_q$, $g(q_l) = b_q$, and $Q_1$ and $Q_2$ are both homomorphically mapped to $P_3$ (i.e., $g(q_r) = b_t$). Observe that $g$ is a homomorphism from $Q$ to $C$ of cost 2. This implies, in particular, that there is a homomorphism from $D'$ to $H$ of cost less than $M$. We now consider three other homomorphisms from $Q$ to $C$:

(a) $f(q_1) = b_1$ and $f(q_l) = b_q$, and $Q_1$ is mapped to $P_4$ and $Q_2$ is mapped to $P_3$ homomorphically (i.e., $f(q_r) = b_t$). The cost of $f$ is 1.

(b) $f'(q_1) = b_q$ and $f'(q_l) = b_1$, and $f'$ maps $Q_1$ to $P_2$ and $Q_2$ to $P_1$ homomorphically (i.e., $f'(q_r) = b_s$). The cost of $f'$ is 1.

(c) $f''(q_1) = b_1$ and $f''(q_l) = b_1$. We will show that the cost of $f''$ is at least $M$. If this is not the case then $f''(q_r) \in \{b_s, b_t\}$ by (ii). First assume that $f''(q_r) = b_s$. By (iii) no vertex of $V(Q_1) - \{q_r\}$ is mapped to $b_2$ and if $s = 2$ then by (iv) $q_{r-1}$ is not mapped to $b_1$. However as $Q_1$ is of the form $(l^+h^+)^{k-1}$ and the path $b_1 b_p b_{p-1} \dots b_s$ is of the form $(l^+h^+)^k$ we get a contradiction to $f''$ mapping $Q_1$ to $C$ and $f''(q_1) = b_1$ and $f''(q_r) = b_s$. So now assume that $f''(q_r) = b_t$. However $Q_2$ is of the form $(l^+h^+)$ and there is no path in $C$ from $b_1$ to $b_t$ of the form $(l^+h^+)$, a contradiction. Therefore the cost of $f''$ is at least $M$.

By the above a minimum cost homomorphism $h: D' \to C$ maps all vertices from a maximum independent set in $D$ to $b_1$ and all other vertices from $D$ to $b_q$. As finding a maximum independent set in a digraph is NP-hard we see that MinHOM($C$) is NP-hard. □

## 5 Dichotomy and Unbalanced Oriented Cycles

We are ready to prove the following main result:

**Theorem 6.** *Let $C$ be an oriented cycle. If $C$ is unbalanced or $C$ is balanced of the form $l^+h^+$ or $C = C_4^0$, then MinHOM($C$) is polynomial-time solvable. Otherwise, MinHOM($C$) is NP-hard.*

By Theorems 4 and 5, to show Theorem 6 it suffices to prove the following:

**Theorem 7.** *Let $C = b_1 b_2 \dots b_p b_1$ be an unbalanced oriented cycle. Then MinHOM($C$) is polynomial-time solvable.*

*Proof.* It is well-known that the minimum cost homomorphism problem to a directed cycle is polynomial-time solvable (see, e.g., [7]). Thus, we may assume that $C$ is not a directed cycle. By Proposition 1, we may assume that level$(b_1) = 0$ and level$(b_i) > 0$ for all $i \in [p] \setminus \{1\}$. Let $q = \max\{j : b_j \in VH(C)\}$.

Consider the oriented path $P = b_1 b_2 \dots b_q$. Let $V_{i+1} = \{b_j \in V(P) : \text{level}(b_j) = i\}$ for all $i \in [k] \cup \{0\}$, where $k = \text{level}(b_q)$. Now consider the oriented path $Q = b_1 b_p b_{p-1} \dots b_q$. Assign levels to the vertices of $Q$ stating from $\text{level}_Q(b_1) = 0$ and continuing as described in Section 1. Observe that all vertices of $Q$ get non-negative levels. Let $U_{i+1} = \{b_j \in V(Q) : \text{level}_Q(b_j) = i\}$, $i \in [l] \cup \{0\}$, where $l = \text{level}_Q(b_q)$. Clearly, $V_1 = U_1 = \{b_1\}$; set $U_{l+1} = V_{k+1}$.

Consider the ordering $\boldsymbol{U} = (b_1, b_p, b_{p-1}, \dots, b_q)$ of the vertices of $Q$. For $i \in [l+1]$, the ordering $\boldsymbol{U}_i$ is obtained from $\boldsymbol{U}$ by deleting all vertices not in $U_i$. Consider the ordering $\boldsymbol{V} = (b_1, b_2, \dots, b_q)$ of the vertices of $P$. For $i \in [k+1]$, the ordering $\boldsymbol{V}_i$ is obtained from $\boldsymbol{V}$ by deleting all vertices not in $V_i$. Observe that the ordering $(\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{U}_2)$ of the vertices of $C[V_1 \cup V_2 \cup U_2]$ has no crossing arcs. Similarly, the ordering $(\boldsymbol{U}_l, \boldsymbol{V}_k, \boldsymbol{V}_{k+1})$ of the vertices of $C[U_l \cup V_k \cup V_{k+1}]$ has no crossing arcs, and orderings $(\boldsymbol{V}_i, \boldsymbol{V}_{i+1})$ and $(\boldsymbol{U}_j, \boldsymbol{U}_{j+1})$ $(i \in [k], j \in [l])$ have no crossing arcs. Thus, $C$ has a $(k+1, l+1)$-ordering of vertices. Now we are done by Theorem 3. □

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