Note on edge-colored graphs and digraphs without properly colored cycles

Gregory Gutin
Department of Computer Science
Royal Holloway, University of London
Egham, Surrey, TW20 0EX, UK
Gutin@cs.rhul.ac.uk

Abstract

We study the following two functions: $d(n,c)$ and $\vec{d}(n,c)$; $d(n,c)$ ($\vec{d}(n,c)$) is the minimum number k such that every c-edge-colored undirected (directed) graph of order n and minimum monochromatic degree (out-degree) at least k has a properly colored cycle. Abouelaoualim et al. (2007) stated a conjecture which implies that $d(n,c)=1$. Using a recursive construction of c-edge-colored graphs with minimum monochromatic degree p and without properly colored cycles, we show that $d(n,c) \geq \frac{1}{c}(\log_c n - \log_c \log_c n)$ and, thus, the conjecture does not hold. In particular, this inequality significantly improves a lower bound on $\vec{d}(n,2)$ obtained by Gutin, Sudakov and Yeo in 1998.

Keywords: edge-colored graphs, properly colored cycles.

1 Introduction

All directed and undirected graphs considered in this paper are simple, i.e., have no loops or parallel edges. We consider only directed cycles in digraphs; the term cycle (in a digraph) will always mean a directed cycle.

Let $G=(V,E)$ be a directed or undirected graph, and let $\chi : E \rightarrow \{1,2,\dots,c\}$ be a fixed (not necessarily proper) edge-coloring of G with c colors, $c \geq 2$. With given χ, G is called a *c-edge-colored* (or, *edge-colored*) graph. A subgraph H of G is called *properly colored* if χ defines a proper edge-coloring of H, i.e., no vertex of H is incident to a pair of edges of the same color. For a vertex of a c-edge-colored graph G, $d_i(x)$ denotes the number of edges of color i incident with x. Let $\delta_{mon}(G) = \min\{d_i(x) : x \in V(G), i \in \{1, 2, \ldots, c\}\}$. If G is directed, $d_i^+(x)$ denotes the number of edges of color i in which x is tail. Let $\delta_{mon}^+(G) = \min\{d_i^+(x) : x \in V(G), i \in \{1, 2, \ldots, c\}\}$.

The authors of [2] stated the following:

Conjecture 1.1 *Let G be a c-edge-colored undirected graph of order n with $\delta_{mon}(G) = d \geq 1$. Then G has a properly colored cycle of length at least* $\min\{n, cd\}$. *Moreover, if $c > 2$, then G has a properly colored cycle of length at least* $\min\{n, cd + 1\}$.

In the next section, using a recursive construction of c-edge-colored graphs with minimum monochromatic degree d and without properly colored cycles, we show that this conjecture does not hold. Moreover, for every $d \geq 1$ there exists an edge-colored graph G with $\delta_{mon}(G) \geq d$ and with no properly colored cycle.

We will study the following two functions: $d(n, c)$ and $\vec{d}(n, c)$; $d(n, c)$ ($\vec{d}(n, c)$) is the minimum number k such that every c-edge-colored graph (digraph) of order n and minimum monochromatic degree (out-degree) at least k has a properly colored cycle. Gutin, Sudakov and Yeo [5] proved the following bounds for $\vec{d}(n, 2)$

$$\frac{1}{4}\log_2 n + \frac{1}{8}\log_2\log_2 n + \Theta(1) \leq \vec{d}(n, 2) \leq \log_2 n - \frac{1}{3}\log_2\log_2 n + \Theta(1) \qquad (1)$$

Using our construction, we prove that $\vec{d}(n, 2) \geq \frac{1}{2}(\log_2 n - \log_2\log_2 n)$. This improves the lower bound in (1). (The lower bound in (1) was obtained using significantly more elaborate arguments.) This bound on $\vec{d}(n, 2)$ follows from lower and upper bounds on $d(n, c)$ and $\vec{d}(n, c)$ obtained for each value of c. The bounds imply that $d(n, c) = \Theta(\log_2 n)$ and $\vec{d}(n, c) = \Theta(\log_2 n)$ for each fixed $c \geq 2$.

Properly colored cycles have been studied in several papers, for a survey, see Chapter 11 in [3]. Properly colored cycles in 2-edge-colored undirected graphs generalize cycles in digraphs and are of interest in genetics [3]. More recent papers on properly colored cycles include [1, 2, 4]. Interestingly, the problem to check whether an edge-colored undirected graph has a properly colored cycle is polynomial time solvable (we can even find a shortest properly colored cycle is polynomial time [1]), but the same problem for edge-colored digraphs is NP-complete [5].

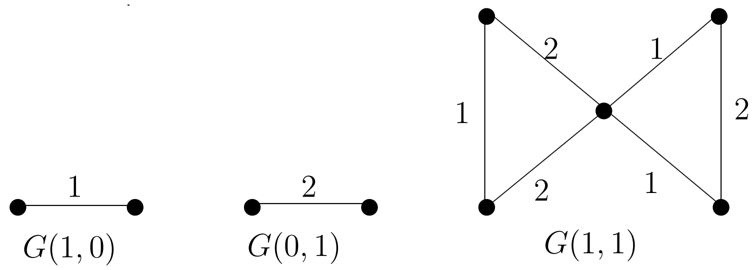

Figure 1: Edge-coloured graphs with no PC cycles.

2 Results

Theorem 2.1 *For each $d \geq 1$ there is an edge-colored graph G with $\delta_{mon}(G) = d$ and with no properly colored cycle.*

Proof: Let $(p_1, p_2, \ldots, p_c)$ be a vector with nonnegative integral coordinates p_i. For an arbitrary $(p_1, p_2, \ldots, p_c)$, $G(p_1, p_2, \ldots, p_c)$ is recursively defined as follows: take a new vertex x and graphs $H_1 = G(p_1 - 1, p_2, p_3, \ldots, p_{c-1}, p_c)$ if $p_1 > 0$, $H_2 = G(p_1, p_2 - 1, p_3, \ldots, p_{c-1}, p_c)$ if $p_2 > 0$, $\ldots$, $H_c = G(p_1, p_2, p_3, \ldots, p_{c-1}, p_c - 1)$ if $p_c > 0$ and add an edge of color i between x and and every vertex of H_i for each i for which $p_i > 0$. In particular, $G(0, 0, \ldots, 0) = K_1$. (Fig. 1 depicts $G(1,0), G(0,1)$ and $G(1,1)$.)

It is easy to see, by induction on $p_1 + p_2 + \cdots + p_c$, that $G = G(p_1, p_2, \ldots, p_c)$ has no properly colored cycle and $\delta_{mon}(G) = \min\{p_i : \ i = 1, 2, \ldots, c\}$. □

In fact, for each $d \geq 1$ there are infinitely many edge-colored graphs G with $\delta_{mon}(G) = d$ and with no properly colored cycle. Indeed, in the construction of $G(p_1, p_2, \ldots, p_c)$ above we may assume that $G(0, 0, \ldots, 0)$ is an edgeless graph of arbitrary order.

Lemma 2.2 *Let $n(p_1, p_2, \ldots, p_c)$ be the order of $G(p_1, p_2, \ldots, p_c)$ and let $n_c(p) = n(p_1, \ldots, p_c)$ for $p = p_1 = \cdots = p_c$. Then $n(p_1, \ldots, p_c) \leq s2^s$, where $s = p_1 + p_2 + \ldots + p_c$, provided $s > 0$ and $p \geq \frac{1}{c}(\log_c n_c(p) - \log_c \log_c n_c(p))$.*

Proof: We first prove $n(p_1, \ldots, p_c) \leq s2^s$ by induction on $s \geq 1$. The inequality clearly holds for $s = 1$. By induction hypothesis, for $s \geq 2$, we have

$$\begin{aligned} n(p_1,\ldots,p_c) &\le 1+\sum_{i=1}^{c}\{n(p_1,\ldots,p_{i-1},p_i-1,p_{i+1},\ldots,p_c):\ p_i>0\} \\ &\le 1+c(s-1)c^{s-1}\le sc^s \end{aligned}$$

Thus, $n_c(p) \le cp\cdot c^{cp}$. Observe that $n_c(p) > ac^a$ provided $a = \log_c n_c(p) - \log_c\log_c n_c(p)$ and, thus, $cp \ge \log_c n_c(p) - \log_c\log_c n_c(p)$. □

Corollary 2.3 *We have* $\vec{d}(n,c) \ge d(n,c) \ge \frac{1}{c}(\log_c n - \log_c\log_c n)$.

Proof: Let H be a c-edge-colored undirected graph and H^* be a digraph obtained from H by replacing every edge $e = xy$ with arcs xy and yx both of color $\chi(e)$. Clearly, H has a properly colored cycle if and only if H^* has a properly colored cycle. Thus, $\vec{d}(n,c) \ge d(n,c)$. The inequality $d(n,c) \ge \frac{1}{c}(\log_c n - \log_c\log_c n)$ follows from Lemma 2.2 and the fact that graphs $G(p,p,\ldots,p)$ have no properly colored cycles. □

We see that $\vec{d}(n,2) \ge \frac{1}{2}(\log_2 n - \log_2\log_2 n)$. This is an improvement over the lower bound on $\vec{d}(n,2)$ in (1). Using the upper bound in (1), we will obtain an upper bound on $\vec{d}(n,c)$ and, thus, $d(n,c)$.

Proposition 2.4 *We have* $\vec{d}(n,c) \le \frac{1}{\lfloor c/2\rfloor}(\log_2 n - \frac{1}{3}\log_2\log_2 n + \Theta(1))$.

Proof: Let D be a c-edge-colored digraph of order n with $\delta_{mon}(D) \ge \frac{1}{\lfloor c/2\rfloor}(\log_2 n - \frac{1}{3}\log_2\log_2 n + \Theta(1))$. Let D' be the 2-edge-colored digraph obtained from D by assigning color 1 to all edges of D of color $1,2,\ldots,\lfloor c/2\rfloor$ and color 2 to all edges of D of color $\lfloor c/2\rfloor+1, \lfloor c/2\rfloor+2,\ldots,c$. It remains to observe that $\delta_{mon}(D') \ge \log_2 n - \frac{1}{3}\log_2\log_2 n + \Theta(1)$ and every properly colored cycle in D' is a properly colored cycle in D. □

Corollary 2.5 *For every fixed* $c \ge 2$*, we have* $d(n,c) = \Theta(\log_2 n)$ *and* $\vec{d}(n,c) = \Theta(\log_2 n)$.

3 Open Problems

We believe that there are functions $s(c), r(c)$ dependent only on c such that $d(n,c) = s(c)\log_2 n(1+o(1))$ and $\vec{d}(n,c) = r(c)\log_2 n(1+o(1))$. In particular, it would be interesting to determine $s(2)$ and $r(2)$.

Acknowledgement This research was supported in part by the IST Programme of the European Community, under the PASCAL Network of Excellence, IST-2002-506778.

References

[1] A. Abouelaoualim, K.Ch. Das, L. Faria, Y. Manoussakis, C.A. Martinhon and R. Saad, Paths and trails in edge-colored graphs. Submitted, 2007.

[2] A. Abouelaoualim, K.Ch. Das, W. Fernandez de la Vega, M. Karpinski, Y. Manoussakis, C.A. Martinhon and R. Saad, Cycles and paths in edge-colored graphs with given degrees. Submitted, 2007.

[3] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London, 2000.

[4] H. Fleischner and S. Szeider, On Edge-Colored Graphs Covered by Properly Colored Cycles. Graphs and Combinatorics 21 (2005), 301–306.

[5] G. Gutin, B. Sudakov and A. Yeo, Note on alternating directed cycles. Discrete Math. 191 (1998), 101-107.