The Complexity of the Minimum Cost Homomorphism Problem for Semicomplete Digraphs with Possible Loops

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Abstract

For digraphs D and H, a mapping $f: V(D)\rightarrow V(H)$ is a homomorphism of D to H if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. For a fixed digraph H, the homomorphism problem is to decide whether an input digraph D admits a homomorphism to H or not, and is denoted as HOM(H).

An optimization version of the homomorphism problem was motivated by a real-world problem in defence logistics and was introduced in [13]. If each vertex $u \in V(D)$ is associated with costs $c_i(u), i \in V(H)$, then the cost of the homomorphism f is $\sum_{u\in V(D)} c_{f(u)}(u)$. For each fixed digraph H, we have the *minimum cost homomorphism problem for* H and denote it as MinHOM(H). The problem is to decide, for an input graph D with costs $c_i(u)$, $u \in V(D), i \in V(H)$, whether there exists a homomorphism of D to H and, if one exists, to find one of minimum cost.

Although a complete dichotomy classification of the complexity of MinHOM(H) for a digraph H remains an unsolved problem, complete dichotomy classifications for MinHOM(H) were proved when H is a semicomplete digraph [10], and a semicomplete multipartite digraph [12, 11]. In these studies, it is assumed that the digraph H is loopless. In this paper, we present a full dichotomy classification for semicomplete digraphs with possible loops, which solves a problem in [9].

1 Introduction, Terminology and Notation

For directed (undirected) graphs G and H, a mapping $f: V(G)\rightarrow V(H)$ is a *homomorphism of* G *to* H if uv is an arc (edge) implies that $f(u)f(v)$ is an arc (edge). A homomorphism f of G to H is also called an H*-coloring* of G, and $f(x)$ is called the *color* of the vertex x in G. We denote the set of all homomorphisms from G to H by $HOM(G, H)$.

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Let H be a fixed directed or undirected graph. The *homomorphism problem*, HOM(H), for H asks whether a directed or undirected input graph G admits a homomorphism to H. The *list homomorphism problem*, ListHOM(H), for H asks whether a directed or undirected input graph G with lists (sets) $L_u \subseteq V(H)$, admits a homomorphism f to H in which $f(u) \in L_u$ for each $u \in V(G)$.

Suppose G and H are directed (or undirected) graphs, and $c_i(u)$, $u \in V(G)$, $i \in V(H)$ are nonnegative *costs*. The *cost of a homomorphism* f of G to H is $\sum_{u \in V(G)} c_{f(u)}(u)$. If H is fixed, the *minimum cost homomorphism problem*, MinHOM(H), for H is the following optimization problem. Given an input graph G, together with costs $c_i(u)$, $u \in V(G)$, $i \in V(H)$, find a minimum cost homomorphism of G to H, or state that none exists.

The minimum cost homomorphism problem was introduced in [13], where it was motivated by a real-world problem in defence logistics. We believe it offers a practical and natural model for optimization of weighted homomorphisms. The problem's special cases include the list homomorphism problem [15, 18] and the general optimum cost chromatic partition problem, which has been intensively studied [14, 19, 20], and has a number of applications [21, 23].

If a directed (undirected) graph G has no loops, we call G *loopless*. If a directed (undirected) graph G has a loop at every vertex, we call G *reflexive*. When we wish to stress that a family of digraphs may contain digraphs with loops, we will speak of digraphs *with possible loops (w.p.l.)* For an undirected graph H, $V(H)$ and $E(H)$ denote its vertex and edge sets, respectively. For a digraph H, $V(H)$ and $A(H)$ denote its vertex and arc sets, respectively.

In this paper, we give a complete dichotomy classification of the complexity of MinHOM(H) when H is a semicomplete digraph with possible loops. A dichotomy of MinHOM(H) when H is a tournament w.p.l. was established in [9], but it is much easier to prove than the more general dichotomy obtained in this paper. A full dichotomy of MinHOM(H) for H being a (general) digraph has not been settled yet and is considered to be a very difficult open problem. Nonetheless, dichotomy have been obtained for special classes of digraphs such as semicomplete digraphs and semicomplete multipartite digraphs; see [10, 11, 12]. All these previous studies, apart from [7, 9], deal only with loopless digraphs.

It is usually assumed when we study the structure of a digraph that it has no loops. This is often a natural assumption since many properties of loopless digraphs can readily be extended to general digraphs w.p.l. as the loops do not affect the important part of the structure of a digraph in the majority of cases. When we investigate homomorphisms of undirected/directed graphs, the situation is different. The homomorphism problem HOM(H) is trivially polynomial time solvable when H has a loop, since we may simply map all the vertices of the input (directed) graph to the vertex having a loop. However, if we wish to get a dichotomy of MinHOM(H) or ListHOM(H), it is not that simple. For example, in [8], it turns out that the class of proper interval graphs is exactly the class of graphs for which MinHOM(H) is polynomial time solvable (assuming, as usual, that $P \neq NP$), provided that H is reflexive. On the other hand, if we assume that H is loopless, MinHOM(H) is polynomial time solvable if and only if H is a proper interval bigraph. It is often the case that, even if we succeeded in obtaining a dichotomy classification of MinHOM(H) for reflexive and loopless H separately, it is an another issue to have a dichotomy classification for H with possible loops.

Complete dichotomy classifications of ListHOM(H) and MinHOM(H), for an undirected graph H w.p.l., have been achieved, see [3, 4, 5] and [8]. For a directed graph with possible loops, the study has just begun and there are only a few results proved [9]. In [6], the authors prove some partial results on complexity of ListHOM(H) when H is a reflexive digraph. Especially, it is conjectured that (unless P=NP) for a reflexive digraph H, ListHOM(H) is polynomial time solvable if and only if H has a proper ordering. Here, we say that a reflexive digraph H has a *proper ordering* if its vertices can be ordered so that whenever $xy, x'y' \in A(H)$, $\min(x, x')\min(y, y')$ is also in $A(H)$. Unfortunately, the conjecture remains unconfirmed even for the case of reflexive semicomplete digraphs.

In the first version of this paper we conjectured that (unless P=NP) for a reflexive digraph H, MinHOM(H) is polynomial time solvable if and only if H has a Min-Max ordering, i.e., its its vertices can be ordered so that whenever $xy, x'y' \in A(H)$, $\min(x, x')\min(y, y')$ and $\max(x, x')\max(y, y')$ are also in $A(H)$. This conjecture was recently proved in [7] using several results, methods and approaches of this paper. Using the main result of [7] one can reduce the length of this paper by shortening Section 2. However, [7] uses some results of Section 2 which therefore must remain in the paper and, in subsequent sections, we use structural results proved in Section 2 but cannot be immediately obtained from the main result of [7]. As a result, the reduction is not significant in terms of paper length. Since we wish to keep the paper self-contained, we have decided not to use the main result of [7] in our paper.

In the rest of this section, we give additional terminology and notation. In the subsequent sections, we first prove a full dichotomy classification of the complexity of MinHOM(H) when H is a reflexive semicomplete digraph. Using this result, we shall further present a full dichotomy classification of MinHOM(H) when H is a semicomplete digraph with possible loops.

For a digraph D, if $xy \in A(D)$, we say that x *dominates* y and y *is dominated* by x, denoted by $x \rightarrow y$. Furthermore, if $xy \in A(D)$ and $yx \notin A(D)$, then we say that x *strictly dominates* y and y *is strictly dominated* by x, denoted by $x \mapsto y$. For sets $X, Y \subseteq V(G)$, $X \rightarrow Y$ means that $x \rightarrow y$ for each $x \in X$, $y \in Y$. Also, for sets $X, Y \subseteq V(G)$, $X \mapsto Y$ means that $xy \in A(D)$ but $yx \notin A(D)$ for each $x \in X$, $y \in Y$. For $xy \in A(D)$, we call xy an *asymmetric arc* if $yx \notin A(D)$, and a *symmetric arc* if $yx \in A(D)$. A digraph D is *symmetric* if each arc of D is symmetric. For a digraph H, H^{sym} denotes the *symmetric subdigraph* of H, i.e., a digraph with $V(H^{sym}) = V(H)$ and $A(H^{sym}) = \{uv, vu \in A(H)\}$. Note that any vertex u of $V(H^{sym})$ has a loop if and only if u has a loop in H. We call a directed graph D an *oriented graph* if all arcs of D are asymmetric.

For a digraph D, let $D[X]$ denote a subdigraph induced by $X \subseteq V(D)$. For any pair of vertices of a directed graph D, we say that u and v are *adjacent* if $u \to v$ or $v \to u$, or both. The *underlying graph* $U(D)$ of a directed graph D is the undirected graph obtained from D by disregarding all orientations and deleting one edge in each pair of parallel edges. A directed graph D is *connected* if $U(D)$ is connected. The *components* of D are the subdigraphs of D induced by the vertices of components of $U(D)$.

By a *directed path (cycle)* we mean a simple directed path (cycle) (i.e., with no self-crossing). We assume that a directed cycle has at least two vertices. In particular, a loop is not a cycle. A directed cycle with k vertices is called a *directed k-cycle* and denoted by $\vec{C}_k$. Let K_n^* denote a complete digraph with a loop at each vertex, i.e., a reflexive complete digraph.

An *empty digraph* is a digraph with no arcs. A loopless digraph D is a *tournament* (*semicomplete digraph*) if there is exactly one arc (at least one arc) between every pair of vertices. We will consider *semicomplete digraphs with possible loops (w.p.l.)*, i.e., digraphs obtained from semicomplete digraphs by appending some number of loops (possibly zero loops). A *k-partite tournaments* (*semicomplete k-partite digraph*) is a digraph obtained from a complete k-partite graph by replacing every edge xy with one of the two arcs xy, yx (with at least one of the arcs xy, yx). An acyclic tournament on p vertices is denoted by TT_p and called a *transitive tournament*. The vertices of a transitive tournament TT_p can be labeled $1, 2, \ldots, p$ such that $ij \in A(TT_p)$ if and only if $1 \leq i < j \leq p$. By TT_p^- $(p \geq 2)$, we denote TT_p without the arc $1p$. For an acyclic digraph H, an ordering $u_1, u_2, \ldots, u_p$ is called *acyclic* if $u_i \to u_j$ implies $i < j$.

Let H be a digraph. The *converse* of H is the digraph obtained from H by replacing every arc xy with the arc yx. For a pair X, Y of vertex sets of a digraph H, we define $X \times Y = \{xy : \ x \in X, y \in Y\}$. Let H be a loopless digraph with vertices $x_1, x_2, \ldots, x_p$ and let $S_1, S_2, \ldots, S_p$ be digraphs. Then the *composition* $H[S_1, S_2, \ldots, S_p]$ is the digraph obtained from H by replacing x_i with S_i for each $i = 1, 2, \ldots, p$. In other words,

$$V(H[S_1, S_2, \ldots, S_p]) = V(S_1) \cup V(S_2) \cup \ldots \cup V(S_p) \text{ and}$$

$$A(H[S_1, S_2, \ldots, S_p]) = \cup\{V(S_i) \times V(S_j) : \ x_i x_j \in A(H), \ 1 \leq i \neq j \leq p\} \cup (\cup_{i=1}^{p} A(S_i)).$$

If every S_i is an empty digraph, the composition $H[S_1, S_2, \ldots, S_p]$ is called an *extension* of H.

The *intersection graph* of a family $F = \{S_1, S_2, \ldots, S_n\}$ of sets is the graph G with $V(G) = F$ in which S_i and S_j are adjacent if and only if $S_i \cap S_j \neq \emptyset$. Note that by this definition, each intersection graph is reflexive. A graph isomorphic to the intersection graph of a family of intervals on the real line is called an *interval graph*. If the intervals can be chosen to be inclusion-free, the graph is called a *proper interval graph.*

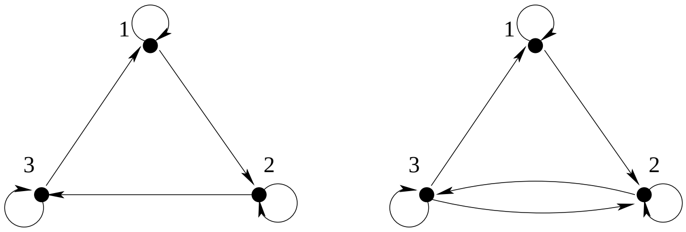

Figure 1: Obstuctions: $\vec{C}_3$ and R

2 Classification for Reflexive Semicomplete Digraphs

In this section, we describe a dichotomy classification of the complexity of MinHOM(H) when H is a reflexive semicomplete digraph. Let R be a reflexive digraph with $V(R) = \{1,2,3\}$ and $A(R) = \{12, 23, 32, 31, 11, 22, 33\}$. Let $\vec{C}_3^*$ denote a reflexive directed cycle on three vertices. (See Figure 1.) The main dichotomy classification of this section is given in the following theorem.

Theorem 2.1 *Let H be a reflexive semicomplete digraph. If H does not contain either R or $\vec{C}_3^*$ as an induced subdigraph, and $U(H^{sym})$ is a proper interval graph (possibly with more than one component), then MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

2.1 NP-hard cases of MinHOM(H)

The following lemma is an obvious basic observation often used to obtain dichotomies. This lemma is certainly applicable for a digraph H w.p.l.

Lemma 2.2 *[10] Let H' be an induced subdigraph of a digraph H. If MinHOMP(H') is NP-hard, then MinHOMP(H) is also NP-hard.*

The following assertion was proved in [9].

Lemma 2.3 *Let a digraph H be obtained from $\vec{C}_k$, $k \geq 3$, by adding at least one loop. Then MinHOM(H) is NP-hard.*

The following lemma shows that for a digraph H obtained from $\vec{C}_3$ by adding some loops and backward arcs, i.e., arcs of the form $(i, i-1)$, MinHOM(H) is NP-hard.

Lemma 2.4 *Let H be a digraph with $V(H) = \{1, 2, 3\}$ and $A(H) = \{12, 23, 32, 31, 22, 33\} \cup B$, where $B \subseteq \{11\}$. Then MinHOM(H) is NP-hard (See Figure 1.)*

Proof: Let G be a loopless digraph with p vertices. Construct a bipartite digraph D as follows: $V(D) = \{x_1, x_2 : x \in V(G)\}$ and $A(D) = \{x_1x_2 : x \in V(G)\} \cup \{x_2y_1 : xy \in A(G)\}$. Set $c_1(x_1) = 0$, $c_2(x_2) = 3$, $c_2(x_1) = c_1(x_2) = 4p + 1$ and $c_3(x_1) = c_3(x_2) = 2$ for each $x \in V(G)$.

Clearly, $h(x_1) = h(x_2) = 3$ for each $x \in V(D)$ defines a homomorphism h of D to H. Let f be a minimum cost homomorphism of D to H. It follows from the fact that the cost of h is $4p$ that $f(x_2) \neq 1$ and $f(x_1) \neq 2$ for each $x \in V(G)$. Thus, for every arc x_1x_2 of D we have three possibilities of coloring: (a) $f(x_1) = 1, f(x_2) = 2$; (b) $f(x_1) = f(x_2) = 3$; (c) $f(x_1) = 3, f(x_2) = 2$. Because of the three choices and the structure of H, if $f(x_1) = 3$ and $f(x_2) = 2$, we can recolor x_2 so that $f(x_2) = 3$, decreasing the cost of f, a contradiction. Thus, (c) is impossible for f.

Let $f(x_1) = f(y_1) = 1$, where x, y are distinct vertices of G. If $xy \in A(G)$, then $x_2y_1 \in A(D)$, which is a contradiction since $f(x_2) = 2$. Thus, x and y are non-adjacent in G. Hence, $I = \{x \in V(G) : f(x_1) = 1\}$ is an independent set in G. Observe that the cost of f is $4p - |I|$.

Conversely, if I is an independent set in G, we obtain a homomorphism g of D to H by fixing $g(x_1) = 1$, $g(x_2) = 2$ for $x \in I$ and $g(x_1) = g(x_2) = 3$ for $x \in V(G) - I$. Observe that the cost of g is $4p - |I|$. Hence a homomorphism g of D to H is of minimum cost if and only if the corresponding independent set I is of maximum size in G. Since the maximum size independent set problem is NP-hard, MinHOM(H) is NP-hard as well. Observe that the validity of the proof does not depend on whether vertex 1 has a loop or not. $\diamond$

Corollary 2.5 *Let H be a reflexive semicomplete digraph. If H contains either R or $\vec{C}_3^*$ as an induced subdigraph, MinHOM(H) is NP-hard.*

Proof: It is straightforward to see that Lemmas 2.3 and 2.4 imply the NP-hardness of MinHOM($\vec{C}_3^*$) and MinHOM(R), respectively. The above statement follows directly from Lemma 2.2. $\diamond$

The following theorem is from [8].

Theorem 2.6 *Let H be a reflexive graph. If H is a proper interval graph (possibly with more than one component), then the problem MinHOM(H) is polynomial time solvable. In all other cases, the problem MinHOM(H) is NP-hard.*

Suppose that H is a semicomplete digraph and that $U(H^{sym})$ is not a proper interval graph. That is, at least one component of $U(H^{sym})$ is not a proper interval graph. Then

MinHOM($U(H^{sym})$) is polynomial time reducible to MinHOM(H) since an input graph G of MinHOM($U(H^{sym})$) can be transfomed into an input digraph G^* of MinHOM(H) by replacing each edge xy of G by a symmetric arc xy of G^*. Hence, if $U(H^{sym})$ is not a proper interval graph, MinHOM(H) is NP-hard by Theorem 2.6. Together with Corollary 2.5, this proves the claim for the NP-hardness part of Theorem 2.1.

Theorem 2.7 *Let H be a reflexive semicomplete digraph. If H contains either R or $\vec{C}_3^*$ as an induced subdigraph, or $U(H^{sym})$ is not a proper interval graph, then MinHOM(H) is NP-hard.*

2.2 Polynomial time solvable cases of MinHOM(H)

Let H be a digraph and let $v_1, v_2, \ldots, v_p$ be an ordering of $V(H)$. Let $e = v_i v_r$ and $f = v_j v_s$ be two arcs in H. The pair $v_{\min\{i,j\}} v_{\min\{s,r\}}$ ($v_{\max\{i,j\}} v_{\max\{s,r\}}$) is called the *minimum* (*maximum*) of the pair e, f. (The minimum (maximum) of two arcs is not necessarily an arc.) An ordering $v_1, v_2, \ldots, v_p$ is a *Min-Max ordering of $V(H)$* if both minimum and maximum of every two arcs in H are in $A(H)$. Two arcs $e, f \in A(H)$ are called a *non-trivial pair* if $\{e, f\} \neq \{g', g''\}$, where g' (g'') is the minimum (maximum) of e, f. Clearly, to check that an ordering is Min-Max, it suffices to verify that the minimum and maximum of every non-trivial pair of arcs are arcs, too.

The following theorem was proved in [10] for loopless digraphs. In fact, the same proof is valid for digraphs with possible loops.

Theorem 2.8 *Let H be a digraph and let an ordering $1, 2, \ldots, p$ of $V(H)$ be a Min-Max ordering, i.e., for any pair ik, js of arcs in H, we have $\min\{i, j\}\min\{k, s\} \in A(H)$ and $\max\{i, j\}\max\{k, s\} \in A(H)$. Then MinHOM(H) is polynomial time solvable.*

In this subsection, we assume that H is a reflexive semicomplete digraph which contains neither R nor $\vec{C}_3^*$, and for which $U(H^{sym})$ is a proper interval graph (possibly with more than one component), unless we mention otherwise. In this subsection, we will show that H has a Min-Max ordering, and, thus, MinHOM(H) is polynomial time solvable by Theorem 2.8.

There is a useful characterization of proper interval graphs [16, 22].

Theorem 2.9 *A reflexive graph H is a proper interval graph if and only if its vertices can be ordered $v_1, v_2, \ldots, v_n$ so that $i < j < k$ and $v_i v_k \in E(H)$ imply that $v_i v_j \in E(H)$ and $v_j v_k \in E(H)$.*

Let H be a digraph and let $v_1, v_2, \ldots, v_p$ be an ordering of $V(H)$. We call $v_i v_j$ a *forward arc* (with respect to the ordering) if $i < j$, and a *backward arc* if $i > j$. The following lemma shows that if H satisfies a certain condition, then the vertices of H can ordered so that every arc is either forward or symmetric.

Lemma 2.10 *Let H be a reflexive semicomplete digraph and suppose H does not contain R as an induced subdigraph and suppose that $U(H^{sym})$ is a connected proper interval graph. Then the vertices of H can be ordered $v_1, v_2, \ldots, v_n$ such that $i < j < k$ and $v_iv_k \in A(H^{sym})$ imply that $v_iv_j \in A(H^{sym})$ and $v_jv_k \in A(H^{sym})$ and furthermore, for every pair of vertices v_i and v_j with $i < j$, we have $v_i \to v_j$.*

Proof: Since $U(H^{sym})$ is a proper interval graph, the vertices of H can be ordered $v_1, \ldots, v_n$ such that $i < j < k$ and $v_iv_k \in A(H^{sym})$ imply that $v_iv_j \in A(H^{sym})$ and $v_jv_k \in A(H^{sym})$ by Theorem 2.9. Observe that if v_iv_j is a symmetric arc with $i < j$, then for each ℓ, k with $i < \ell < k < j$ we have $v_\ell v_k$ is a symmetric arc. Note also that v_iv_{i+1} for each $i = 1, \ldots, n-1$ is a symmetric arc, since otherwise H^{sym} has more that one component, contradicting the connectivity assumption.

We wish to prove that if $v_\ell \to v_k$ for some $\ell < k$, then $v_i \to v_j$ for each $i < j$. We prove it using a sequence of claims.

Claim 1. If $v_i \mapsto v_j$ for some $j > i$, then $v_i \to \{v_{i+1}, \ldots, v_n\}$.

Proof: If $v_iv_k \in A(H^{sym})$ for each $k > i$, there is nothing to prove. Thus, we may assume without loss of generality that there exists a vertex v_k such that $v_k \mapsto v_i$. By an observation above, all arcs between v_i and v_t for each $t \geq \min\{j, k\}$ are asymmetric. Thus, there is an index $m \geq \min\{j, k\}$ such that either $v_i \mapsto v_m$ and $v_{m+1} \mapsto v_i$ or $v_m \mapsto v_i$ and $v_i \mapsto v_{m+1}$. Recall that that v_mv_{m+1} is a symmetric arc. Hence, $H[\{v_i, v_m, v_{m+1}\}] \cong R$, a contradiction.

A similar argument leads to the symmetric statement below.

Claim 1′. If $v_j \mapsto v_i$ for some $j < i$, then $\{v_1, \ldots, v_{i-1}\} \to v_i$.

Claim 2. We have either $\{v_1, \ldots, v_{i-1}\} \to v_i \to \{v_{i+1}, \ldots, v_n\}$ or $\{v_{i+1}, \ldots, v_n\} \to v_i \to \{v_1, \ldots, v_{i-1}\}$.

Proof: Suppose to the contrary that there are two vertices v_j, v_k with $j < i < k$ such that $v_i \mapsto v_j$, $v_i \mapsto v_k$ in H. (The case for which $v_j \mapsto v_i$, $v_k \mapsto v_i$ in H can be treated in a similar manner.) Then v_jv_k is not a symmetric arc since otherwise, v_jv_i and v_iv_k must be symmetric arcs by the property of the ordering. Hence, only one of v_jv_k and v_kv_j is an arc of H. In either case, we have a contradiction by Claims 1 or 1′.

Claim 3: If $v_\ell \to v_k$ for some $\ell < k$, then $v_i \to v_j$ for each $i < j$.

Proof: Suppose to the contrary that there exist two vertices v_j and v_i such that $v_j \mapsto v_i$ and $i < j$. If any two of the four vertices v_i, v_j, v_ℓ and v_k are identical, we have a contradiction by Claim 1, 1′ or 2. Thus, we may assume that these vertices are all distinct. We have the following cases.

(a) Let $i < \ell$. Then we have $i < \ell < k$. If v_iv_k is a symmetric arc, the arc $v_\ell v_k$ must be symmetric by the property of the ordering, a contradiction. Hence, only one of v_iv_k and v_kv_i is an arc of H. If $v_i \mapsto v_k$, we have a contradiction by Claim 1 for vertex v_i and if $v_k \mapsto v_i$, we have a contradiction by Claim 1′ for vertex v_k.

(b) Let $\ell < i$. Then we have $\ell < i < j$. If $v_\ell v_j$ is a symmetric arc, then v_iv_j must be a symmetric arc by the property of the ordering, contradiction. Hence, only one of v_jv_ℓ and $v_\ell v_j$ is an arc of H. In either case, we have a contradiction by Claims 1 or 1′.

By Claim 3, either $v_1, v_2, \ldots, v_n$ or its reversal satisfies the required property. $\diamond$

Consider a reflexive semicomplete digraph H. Suppose that H does not contain either R or $\vec{C}_3^*$ as an induced subdigraph and $U(H^{sym})$ is a proper interval graph. Note that each isolated vertex in $U(H^{sym})$ forms a trivial proper interval graph in itself.

Suppose that H^{sym} is not connected. If each component of H^{sym} is trivial, it is clear that H has a Min-Max ordering since H is a reflexive transitive tournament (H does not contain $\vec{C}_3^*$). Hence we may assume that at least one component of H^{sym} is nontrivial. Let H_i^{sym} and H_j^{sym} be two distinct components of H^{sym} and at least one of them, say H_j^{sym}, is a nontrivial component containing more than one vertex. Clearly, the arcs between H_i^{sym} and H_j^{sym} are all asymmetric.

Let u be a vertex of H_i^{sym}, and let v and w be two distinct vertices in H_j^{sym}. Without loss of generality, we may assume that $u \mapsto v$. If $w \mapsto u$, there must exist adjacent vertices p and q on a path from v to w in H_j^{sym} such that $u \mapsto p$ and $q \mapsto u$. Then we have $H[\{u, p, q\}] \cong R$, a contradiction. With a similar argument, it is easy to see that all arcs between two components H_i^{sym} and H_j^{sym} are oriented in the same direction with respect to the components. Furthermore, since H is $\vec{C}_3^*$-free, the components of H^{sym} can be ordered $H_1^{sym}, H_2^{sym}, \ldots, H_l^{sym}$ so that for each pair of vertices $u \in H_i^{sym}$ and $v \in H_j^{sym}$ with $i < j$, we have $u \mapsto v$. This implies the following:

Corollary 2.11 *Let H be a reflexive semicomplete digraph and suppose H does not contain either R or $\vec{C}_3^*$ as an induced subdigraph and suppose that $U(H^{sym})$ is a proper interval graph. Then the components of H^{sym} can be ordered $H_1^{sym}, H_2^{sym}, \ldots, H_l^{sym}$ such that if $u \in H_i^{sym}$, $v \in H_j^{sym}$ and $i < j$, then we have $u \mapsto v$.*

We shall call the ordering of the components of H^{sym} described in Corollary 2.11 an *acyclic ordering* of the the components of H^{sym}. Now, with Lemma 2.10, we have the following lemma.

Lemma 2.12 *Let H be a reflexive semicomplete digraph and suppose H does not contain either R or $\vec{C}_3^*$ as an induced subdigraph and $U(H^{sym})$ is a proper interval graph. Then the vertices of H can be ordered $v_1, \ldots, v_n$ so that $i < j < k$ and $v_iv_k \in A(H^{sym})$ imply* *that $v_iv_j \in A(H^{sym})$ and $v_jv_k \in A(H^{sym})$ and furthermore, for every pair of vertices v_i and v_j from $V(H)$ with $i < j$, we have $v_i \to v_j$.*

Proof: Let $H_1^{sym}, H_2^{sym}, \ldots, H_l^{sym}$ be the acyclic ordering of the components of H^{sym}. By Lemma 2.10, we have an ordering $v_1^i, v_2^i, \ldots, v^i_{|V(H_i^{sym})|}$ of $V(H_i^{sym})$, for each $i = 1, \ldots, l$, such that every asymmetric arc is forward. Then the ordering

$$v_1^1, v_2^1, \ldots, v^1_{|V(H_1^{sym})|}, v_1^2, v_2^2, \ldots, v^2_{|V(H_2^{sym})|}, \ldots, v_1^l, v_2^l, \ldots, v^l_{|V(H_l^{sym})|}$$

of the vertices of H satisfies the condition, completing the proof. ⋄

The following proposition was proved in [8]. Observe that for the symmetric subdigraph H^{sym}, the ordering of the vertices of H described in Lemma 2.12 satisfies the condition of the proposition below.

Proposition 2.13 *A reflexive graph H has a Min-Max ordering if and only if its vertices can be ordered $v_1, v_2, \ldots, v_n$ so that $i < j < k$ and $v_iv_k \in E(H)$ imply that $v_iv_j \in E(H)$ and $v_jv_k \in E(H)$.*

Lemma 2.14 *Let H be a reflexive semicomplete digraph. If H does not contain either R or $\vec{C}_3^*$ as an induced subdigraph, and $U(H^{sym})$ is a proper interval graph, then H has a Min-Max ordering.*

Proof: Let $v_1, \ldots, v_n$ be the ordering of $V(H)$ as described in Lemma 2.12. We will show that this is a Min-Max ordering of $V(H)$.

Let v_iv_j and v_kv_l be a non-trivial pair of H. If $i \leq j$ and $k \leq l$, it is easy to see that both the minimum and maximum of v_iv_j and v_kv_l are in $A(H)$. If $i > j$ and $k > l$, v_iv_j and v_kv_l are symmetric arcs of H^{sym}. Since they are a non-trivial pair, the vertices v_i, v_j, v_k and v_l belong to the same component of H^{sym} by the proof of Lemma 2.12. Then the minimum and the maximum of v_iv_j and v_kv_l are also in $A(H^{sym})$ by Lemma 2.10.

Now suppose that $i \leq j$ and $k > l$. Note that v_kv_l is a symmetric arc. Hence if $i = j$, then the vertices v_i, v_k and v_l belong to the same component of H^{sym}, and, thus, the minimum and the maximum of v_iv_j and v_kv_l are in $A(H^{sym})$ by Lemma 2.10. If $i \neq j$, we need to consider the following four cases covering all possibilities for non-trivial pairs:

(a) $i \leq l < j \leq k$. Then $v_{\min\{i,k\}}v_{\min\{j,l\}} = v_iv_l \in A(H)$ as $i \leq l$. Also, since v_lv_k is a symmetric arc, $v_{\max\{i,k\}}v_{\max\{j,l\}} = v_kv_j$ is a symmetric arc by Lemma 2.12.

(b) $l < i < j \leq k$. Then, since v_lv_k is a symmetric arc, $v_{\max\{i,k\}}v_{\max\{j,l\}} = v_kv_j$ and $v_{\min\{i,k\}}v_{\min\{j,l\}} = v_iv_l$ are symmetric arcs by Lemma 2.12.

(c) $l < i < k < j$. Then, $v_{\max\{i,k\}}v_{\max\{j,l\}} = v_kv_j \in A(H)$ as $k < j$. Also, since v_lv_k is a symmetric arc, $v_{\min\{i,k\}}v_{\min\{j,l\}} = v_iv_l$ is a symmetric arc by Lemma 2.12.

(d) $i \leq l < k < j$. Then $v_{\min\{i,k\}}v_{\min\{j,l\}} = v_iv_l \in A(H)$ and $v_{\max\{i,k\}}v_{\max\{j,l\}} = v_kv_j \in A(H)$ as $i \leq l$ and $k < j$. ⋄

Theorem 2.15 *Let H be a reflexive semicomplete digraph. If H does not contain either R or $\vec{C}_3^*$ as an induced subdigraph, and $U(H^{sym})$ is a proper interval graph, then MinHOM(H) is polynomial time solvable.*

Proof: This is a direct consequence of Lemmas 2.14 and 2.8. ⋄

Corollary 2.16 *Suppose $P \neq NP$. Let H be a reflexive semicomplete digraph. Then MinHOM(H) is polynomial time solvable if and only if H has a Min-Max ordering.*

Proof: This is a direct consequence of Lemma 2.14 and Theorem 2.7.

3 Classification for Semicomplete Digraphs with Possible Loops

In this section, we describe a dichotomy classification for MinHOM(H) when H is a semicomplete digraph with possible loops. Let W be a digraph with $V(W) = \{1, 2\}$ and $A(W) = \{12, 21, 22\}$. Let R' be a digraph with $V(R') = \{1, 2, 3\}$ and $A(R') = \{12, 23, 32, 31, 22, 33\}$. (See Figure 2)

Given a semicomplete digraph H w.p.l., let $L = L(H)$ and $I = I(H)$ denote the maximal induced subdigraphs of H which are reflexive and loopless, respectively. When $H = L$, we have obtained a dichotomy classification for reflexive semicomplete digraph in Section 2. When $H = I$, we also have a dichotomy classification by the following theorem from [10].

Theorem 3.1 *For a semicomplete digraph H, MinHOM(H) is polynomial time solvable if H is acyclic or $H = \vec{C}_k$ for k =2 or 3, and NP-hard, otherwise.*

In this section, we will show that the following dichotomy classification holds when H is a semicomplete digraph w.p.l.

Theorem 3.2 *Let H be a semicomplete digraph with possible loops. If one of the following holds, then MinHOM(H) is polynomial time solvable. Otherwise, it is NP-hard.*

(i) The digraph $H = \vec{C}_k$ for k =2 or 3.

(ii-a) The digraph L does not contains either R or $\vec{C}_3^*$ as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph; I is a transitive tournament; H does not contain either W, R' or $\vec{C}_3$ with at least one loop as an induced subdigraph.

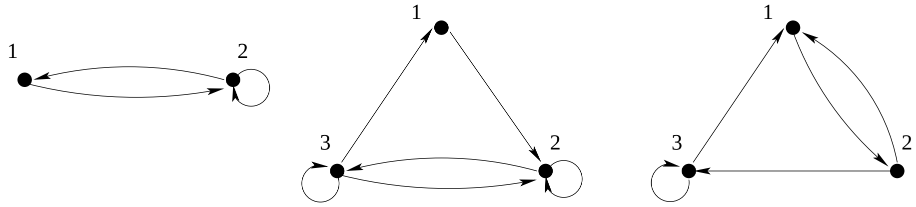

Figure 2: Obstuctions: W, R' and the obstruction from Lemma 3.6

or equivalently,

(ii-b) The digraph $H = TT_k[S_1, S_2, \ldots, S_k]$ where S_i for each $i = 1, \ldots, k$ is either a single vertex without a loop, or a reflexive semicomplete digraph which does not contain R as an induced subdigraph and for which $U(S_i^{sym})$ is a connected proper interval graph.

Through subsections 3.1 and 3.2, we will consider only the polynomiality condition (ii-a) in Theorem 3.2. We first prove the NP-hardness part of Theorem 3.2 in subsection 3.1. In subsection 3.2, a proof for the polynomial solvable case is given. Finally in subsection 3.3, we will prove the equivalence of condition (ii-a) and (ii-b) in Theorem 3.2.

3.1 NP-hard cases of MinHOM(H)

The following two lemmas were proved in [9].

Lemma 3.3 *MinHOM(W) is NP-hard.*

Lemma 3.4 *Let H' be a digraph obtained from $\vec{C}_k = 12\ldots k1$, $k \geq 2$, by adding an extra vertex $k+1$ dominated by at least two vertices of the cycle and let H'' is the digraph obtained from H' by adding the loop at vertex $k+1$. Let H be H' or its converse or H'' or its converse. Then MinHOM(H) is NP-hard.*

Observe that MinHOM(R') is NP-hard by Lemma 2.4. The following result was proved in [2].

Theorem 3.5 *Let H be a (loopless) semicomplete digraph with at least two directed cycles. Then the problem of checking whether a digraph D has an H-coloring is NP-complete.*

Lemma 3.6 *Let H be a digraph with $V(H) = \{1, 2, 3\}$ and $A(H) = \{12, 21, 23, 31, 33\}$. Then MinHOM($H$) is NP-hard. (See Figure 2)*

Proof: We will reduce the maximum independent set problem to MinHOM(H). Before we do this we consider a digraph $D^*(u,v)$ defined as follows. Here we set $e = uv$:

$$V(D^*(u,v)) = \{u^e, u, v^e, v, x_1^e, x_2^e, \dots, x_6^e\}$$

$$A(D^*(u,v)) = \{x_1^e x_2^e, x_2^e x_3^e, \dots, x_5^e x_6^e, x_6^e x_1^e, x_4^e u^e, u^e u, x_5^e v^e, v^e v\}$$

Let G be a graph with p vertices. Construct a digraph D as follows: Start with $V(D) = V(G)$ and, for each edge $e = uv \in E(G)$, add a distinct copy of $D^*(u,v)$ to D. Note that the vertices in $V(G)$ form an independent set in D and that $|V(D)| = |V(G)| + 8|E(G)|$.

Given an edge $e = uv \in E(G)$, we fix the costs as follows: Let $c_1(x_1^e) = 0$ and $c_i(x_1^e) = p+1$ for each $i = 2,3$. Let $c_i(x_j^e) = 0$ for each $i = 1,2,3$ and $j = 2,\dots,6$ apart from $c_3(x_4^e) = c_3(x_5^e) = p+1$. Also, $c_i(u^e) = c_i(v^e) = 0$ for each $i = 1,2,3$, $c_2(u) = c_2(v) = 0$, $c_1(u) = c_1(v) = 1$ and $c_3(u) = c_3(v) = p+1$.

Consider a mapping h of $V(D)$ to $V(H)$ as follows: $h(x_i^e) = 1$ if i is odd, $h(x_i^e) = 2$ if i is even, $h(u^e) = 3$, $h(v^e) = 2$ for each $e \in E(G)$ and $h(u) = 1$ for each $u \in V(G)$. It is easy to check that h defines a homomorphism of D to H and the cost of h is p. Let f be a minimum cost homomorphism of D to H. It follows from the fact that the cost of h is p that $f(x_1^e) = 1$, $f(x_4^e), f(x_5^e) \in \{1,2\}$ for each $e \in E(G)$, and $f(u) \in \{1,2\}$ for each each $u \in V(G)$. Moreover, due to the structure of $D^*(u,v)$ and the costs, for each $e \in E(G)$, $(f(x_1^e), \dots, f(x_6^e))$ must coincide with one of the following two sequences: (1,2,1,2,1,2) or (1,2,3,1,2,3).

If the first sequence is the actual one, then we have $f(x_4^e) = 2$, $f(u^e) \in \{1,3\}$, $f(u) \in \{1,2\}$ and $f(x_5^e) = 1$, $f(v^e) = 2$, $f(v) = 1$. If the second sequence is the actual one, we have $f(x_4^e) = 1$, $f(u^e) = 2$, $f(u) = 1$ and $f(x_5^e) = 2$, $f(v^e) \in \{1,3\}$, $f(v) \in \{1,2\}$. So in both cases we can assign both u and v color 1. Furthermore, by choosing the right sequence we can color one of u and v with color 2 and the other with color 1. Notice that f cannot assign color 2 to both u and v.

Clearly, f must assign as many vertices of $V(G)$ in D color 2. However, if uv is an edge in G, by the argument above, f cannot assign color 2 to both u and v. Hence, $I = \{u \in V(G) : \; f(u) = 2\}$ is an independent set in G. Observe that the cost of f is $p - |I|$.

Conversely, if I' is an independent set in G, we obtain a homomorphism g of D to H by fixing $g(u) = 2$ for $u \in I'$, $g(u) = 1$ for $u \notin I'$. We can choose an appropriate sequence for $x_1^e, \dots, x_6^e$ for each edge $e \in E(G)$ and fix the assignment of u^e and v^e accordingly by the above argument. Observe that the cost of g is $p - |I'|$. Hence the cost of a minimum homomorphism f of D to H is $p - \alpha$, where α is the size of the maximum independent set in G. Since the maximum size independent set problem is NP-hard, MinHOM(H) is NP-hard as well. $\diamond$

Lemma 3.7 *Let H be a digraph with $V(H) = \{1,2,3\}$ and $A(H) = \{12,21,23,31\} \cup B_1 \cup B_2$, where B_1 is either $\{11,22\}$ or $\emptyset$ and B_2 is either $\{33\}$ or $\emptyset$. Then MinHOM(H) is NP-hard.*

Proof: Consider the following three cases.

Case 1: $B_1 = \emptyset$ and $B_2 = \emptyset$. Then MinHOM(H) is NP-hard by Theorem 3.5.

Case 2: $B_1 = \emptyset$ and $B_2 = \{33\}$. Then MinHOM(H) is NP-hard by Lemma 3.6

Case 3: $B_1 = \{11,22\}$. Then MinHOM(H) is NP-hard by Lemma 2.4. ◇

Consider a strong semicomplete digraph w.p.l. H on three vertices. We want to obtain all polynomial cases for H. If H does not have a 2-cycle, MinHOM(H) is NP-hard if (and only if) at least one of its vertices has a loop by Lemma 2.3. Note that we have a polynomial case if H is $\vec{C}_3$. Suppose that H has at least one 2-cycle. If there are two or more 2-cycles, MinHOM(H) is NP-hard by Theorem 3.5 and Lemma 3.3 unless H is reflexive. Note that in the reflexive case, MinHOM(H) is polynomial time solvable since H has a Min-Max ordering.

Now suppose that H has only one 2-cycle. For MinHOM(H) to be not NP-hard, both or neither of the two vertices forming the 2-cycle must have loops simultaneously since otherwise, MinHOM(H) is NP-hard by Lemma 3.3. Now by Lemma 3.7, MinHOM(H) is still NP-hard.

Let $K_3^* - e$ be a digraph obtained by removing a nonloop arc from K_3^*. The above observation can be summarized by the following statement.

Corollary 3.8 *Let H be a strong semicomplete digraph w.p.l. on three vertices. If H is either $\vec{C}_3$, K_3^* or $K_3^* - e$, MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

For a semicomplete digraph w.p.l. H, if either MinHOM(L) or MinHOM(I) is NP-hard, MinHOM(H) is NP-hard by Lemma 2.2. (Recall that $L = L(H)$ and $I = I(H)$ denote the maximal induced subdigraphs of H which are reflexive and loopless, respectively.) Also, if H contains $\vec{C}_3$ with at least one loop as an induced subdigraph, MinHOM(H) is NP-hard by Lemmas 2.3 and 2.2. Suppose MinHOM(H) is not NP-hard. Then Lemma 3.3 indicates that for any pair of vertices $u \in V(L)$ and $v \in V(I)$, either $u \rightarrow v$ or $v \rightarrow u$, not both, as otherwise MinHOM(H) is NP-hard. With these observations and Lemma 3.4, the following statement is easily derived.

Lemma 3.9 *Let H be a semicomplete digraph. If one of the following condition holds, MinHOM(H) is NP-hard.*

(a) I contains a cycle and $I \neq \vec{C}_k$ for k =2 or 3.

(b) L contains either R or $\vec{C}_3^*$ as an induced subdigraph, or $U(L^{sym})$ is not a proper interval graph.

(c) $I = \vec{C}_k$ for k =2 or 3, and L is nonempty.

(d) H contains W, R' or $\vec{C}_3$ with at least one loop as an induced subdigraph.

Proof: If condition (a) holds, MinHOM(H) is NP-hard by Theorem 3.1 and Lemma 2.2. If condition (b) holds, MinHOM(H) is NP-hard by Theorem 2.1 and Lemma 2.2. If condition (d) holds, MinHOM(H) is NP-hard by Lemmas 3.3, 2.4, 2.3 and 2.2. The only remaining part is to prove that the condition (c) is sufficient for MinHOM(H) to be NP-hard.

If $I = \vec{C}_2$ and u is a vertex of L, then we may assume that either u dominates both vertices of I, or u is dominated by one of $V(I)$ and dominates the other without loss of generality. In the former case, MinHOM(H) is NP-hard by Lemma 3.4. In the latter case, MinHOM(H) is NP-hard by Lemma 3.6. If $I = \vec{C}_3$ and u is a vertex with a loop, MinHOM(H) is NP-hard by Lemma 3.4. ◇

In fact, Lemma 3.9 proves the NP-hardness part in Theorem 3.2. This can be seen as follows. Suppose that MinHOM(H) is not NP-hard. Recall that the polynomiality conditions of Theorem 3.2 are: (i) $H = \vec{C}_k$ for k =2 or 3, or (ii-a) L does not contain either R or $\vec{C}_3^*$ as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph, I is a transitive tournament, and H does not contain either W, R' or $\vec{C}_3$ with at least one loop as an induced subdigraph.

Suppose that a semicomplete digraph w.p.l. H has an loopless cycle. Then condition (ii-a) does not hold, and for condition (i) to be violated, either one of (a) and (c) in Lemma 3.9 must hold. On the other hand, suppose that the loopless part I of H is a transitive tournament. Then condition (i) does not hold, and for condition (ii-a) to be violated, one of (b) and (d) in Lemma 3.9 must hold.

Corollary 3.10 *Let H be a semicomplete digraph with possible loops. If none of the following holds, then MinHOM(H) is NP-hard.*

(a) The digraph $H = \vec{C}_k$ for k =2 or 3.

(b) The digraph L does not contains either R or $\vec{C}_3^*$ as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph; I is a transitive tournament; H does not contain either W, R' or $\vec{C}_3$ with at least one loop as an induced subdigraph.

3.2 Polynomial time solvable cases of MinHOM(H)

If condition (i) in Theorem 3.2 holds for a semicomplete digraph w.p.l. H, MinHOM(H) is clearly polynomial time solvable by Theorem 3.1. Although $\vec{C}_3$ does not have a Min-Max ordering, there is a simple algorithm which solves MinHOM(H) in polynomial time when $H = \vec{C}_k$, $k \geq 2$, see [10, 9].

The equivalence of the conditions (ii-a) and (ii-b) will be shown in the last subsection. Therefore, we only need to prove that when H satisfies the condition (ii-a) in Theorem 3.2, MinHOM(H) is polynomial time solvable. We claim that H has a Min-Max ordering in this case. Before showing this claim, we prove that the ordering described in Lemma 2.12 for a reflexive semicomplete digraph can be extended to a semicomplete digraph w.p.l. if condition (ii-a) in Theorem 3.2 is satisfied.

Lemma 3.11 *Let H be a semicomplete digraph with possible loops. Suppose that L does not contain either R or $\vec{C}_3^*$ as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph. Also suppose that I is a transitive tournament and H does not contain either W, R' or $\vec{C}_3$ with at least one loop as an induced subdigraph. Then the vertices of H can be ordered $v_1, \ldots, v_n$ so that for every pair of vertices v_i and v_j with $i < j$, we have $v_i \rightarrow v_j$.*

Proof: Let $L_1^{sym}, \ldots, L_l^{sym}$ be the acyclic ordering of the components of L^{sym}. Let

$$w_1, w_2, \ldots, w_q = v_1^1, v_2^1, \ldots, v^1_{|V(L_1^{sym})|}, \ldots, v_1^i, v_2^i, \ldots, v^i_{|V(L_i^{sym})|}, \ldots, v_1^l, v_1^l, \ldots, v^l_{|V(L_l^{sym})|}$$

be the ordering of $V(L)$ as described in Lemma 2.12. Let $u_1, \ldots, u_p$ be the acyclic ordering of $V(I)$, i.e., $u_i \rightarrow u_j$ implies $i < j$. We will prove the statement by showing that the subdigraph induced by $V(L_i^{sym})$ can be 'inserted' into an appropriate position among the acyclic ordering of $V(I)$ without creating a cycle, thus by constructing an ordering of $V(H)$ satisfying the asserted property.

Observe that any arc whose end vertices are from I and L each is asymmetric. Otherwise, the end vertices of such arch induce a digraph W, a contradiction.

First, we claim that given a vertex u of I and a component L_i^{sym}, we have either $u \mapsto V(L_i^{sym})$ or $V(L_i^{sym}) \mapsto u$. If L_i^{sym} is a trivial component consisting of a single vertex, the claim follows directly. So, assume that $|V(L_i^{sym})| \geq 2$ and there exists two vertices v, v' of L_i^{sym} such that $u \mapsto v$ and $v' \mapsto u$. Then, since L_i^{sym} is connected, there is a path in L_i^{sym} linking v and v'. We can find two adjacent vertices s, t on this path such that $u \mapsto s$ and $t \mapsto u$. However, then $H[\{u, s, t\}] \cong R'$, a contradiction.

Secondly, we claim that for each component L_i^{sym} (possibly trivial), the vertices of L_i^{sym} can be 'inserted' into an appropriate position so that the ordering of $V(I) \cup V(L_i^{sym})$ satisfies the required property. (Here, the ordering within $V(L_i^{sym})$ remains unchanged.) That is, either $V(I) \mapsto V(L_i^{sym})$ or $V(L_i^{sym}) \mapsto V(I)$, or there exist an integer $1 \leq j < p$ such that for all $k \leq j$, we have $u_k \mapsto V(L_i^{sym})$ and for all $k > j$, we have $V(L_i^{sym}) \mapsto u_k$. If $V(I) \mapsto V(L_i^{sym})$ or $V(L_i^{sym}) \mapsto V(I)$, the ordering $u_1, \ldots, u_p$ followed by or following the ordering of $V(L_i^{sym})$ trivially satisfies the required property. Thus, we may assume that $u \mapsto V(L_i^{sym})$ and $V(L_i^{sym}) \mapsto u'$ for some $u, u' \in V(I)$.

Suppose that there are two vertices u_j and $u_{j'}$ of I with $j' < j$ such that $u_j \mapsto V(L_i^{sym})$ and $V(L_i^{sym}) \mapsto u_{j'}$. Then $u_{j'}, u_j$ together with a vertex of L_i^{sym} form $\vec{C}_3$ with a loop, contradicting the assumption. Hence, if $u_j \mapsto V(L_i^{sym})$ for some $u_j \in V(I)$, then $u_{j'} \mapsto V(L_i^{sym})$ for each $j' < j$. Similarly, if $V(L_i^{sym}) \mapsto u_j$ for some $u_j \in V(I)$, then $V(L_i^{sym}) \mapsto u_{j'}$ for each $j' > j$. By taking the maximum j such that $u_j \mapsto V(L_i^{sym})$

and inserting $V(L_i^{sym})$ between u_j and u_{j+1} while preserving the ordering within $V(L_i^{sym})$, we are done with the claim. From now on, we will say that $V(L_i^{sym})$ *is inserted after* u_j if $u_j \mapsto V(L_i^{sym})$ and $V(L_i^{sym}) \mapsto u_{j+1}$ when u_{j+1} exists, and $V(L_i^{sym})$ *is inserted before* u_j if $V(L_i^{sym}) \mapsto u_j$ and $u_{j-1} \mapsto V(L_i^{sym})$ when u_{j-1} exists.

Note that if any two components L_i^{sym} and L_j^{sym} of L^{sym} are inserted before/after the same vertex of I, we will keep their relative order unchanged.

Now let us show that the insertion of all $V(L_i^{sym})$'s does not change their relative order in L. That is, if $V(L_i^{sym})$ is inserted after u_j, then each component $L_{i'}^{sym}$ for $i' > i$ is inserted after $u_{j'}$, where $j' \geq j$ and if L_i^{sym} is inserted before u_j, then each component $L_{i'}^{sym}$ for $i' < i$ is inserted before $u_{j'}$, where $j' \leq j$.

Suppose to the contrary that there are two components L_i^{sym} and $L_{i'}^{sym}$ with $i < i'$ such that $V(L_i^{sym})$ is inserted after u_j and $V(L_{i'}^{sym})$ is inserted before $u_{j'}$ with $j' \leq j$. Then, by the above argument, $V(L_{i'}^{sym}) \mapsto u_j$. However, a vertex from $V(L_i^{sym})$, a vertex from $V(L_{i'}^{sym})$ and u_j induce $\vec{C}_3$ with two loops, contradicting the assumption. Hence, if $V(L_i^{sym})$ is inserted after u_j, then $V(L_{i'}^{sym})$ for $i' > i$ is inserted after $u_{j'}$, where $j' \geq j$. Similarly, we can show that if $V(L_i^{sym})$ is inserted before u_j, then $V(L_{i'}^{sym})$ for $i' < i$ is inserted before $u_{j'}$, where $j' \leq j$.

It is straightforward from the above construction that the resulting ordering satisfies the required property. ◇

Now we are ready to prove that H has a Min-Max ordering when H satisfies the condition (ii-a) in Theorem 3.2.

Lemma 3.12 *Let H be a semicomplete digraph with possible loops. Suppose that L contains neither R nor $\vec{C}_3^*$ as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph. Also suppose that I is a transitive tournament and H does not contain either W, R' or $\vec{C}_3$ with at least one loop as an induced subdigraph. Then MinHOM(H) has a Min-Max ordering.*

Proof: Consider an ordering $v_1, \ldots, v_n$ of the vertices of H as described in Lemma 3.11. We will show that this is a Min-Max ordering of $V(H)$. Note that the induced ordering of $V(I)$ is an acyclic ordering and the induced ordering of $V(L)$ is a Min-Max ordering for L^{sym} as described in Lemma 2.12.

Let v_iv_j and v_kv_l be any nontrivial pair of arcs of H. Observe that if both arcs are in $A(L)$, then the minimum and the maximum of them are also in $A(L)$ since the induced ordering of $V(L)$ is a Min-Max ordering for L. Moreover, if both arcs are forward arcs, i.e. $i < j$ and $k < l$, then we have either $i < k < l < j$ or $k < i < j < l$. In either case, it follows from Lemma 3.11 that the minimum and the maximum of them are in $A(H)$.

Hence what we need to consider is the case where $v_k v_l$ is not a forward arc. If $v_k v_l$ is a loop, then $i < k = l < j$. It follows from Lemma 3.11 that the minimum and the maximum of the two arcs are in $A(H)$ in this case. Let $v_k v_l$ be a backward arc, i.e., $k > l$. Clearly, $v_k v_l \in A(L)$. Then there are two remaining cases to consider.

Case 1: $v_i v_j \in A(I)$.

Then we have one of the following options: (a) $i < l < j < k$, (b) $i < l < k < j$, (c) $l < i < k < j$, (d) $l < i < j < k$. However, in (a), $v_l \mapsto v_j$ and $v_j \mapsto v_k$, which is a contradiction since v_k, v_l belong to the same component of L^{sym}, and v_j has to either dominate or to be dominated by each component of L^{sym}. With a similar argument, case (c) and case (d) are impossible. By Lemma 3.11, in case (b), the minimum and the maximum of $v_i v_j$ and $v_k v_l$ are in $A(H)$.

Case 2: $v_i v_j \in A(H) \setminus (A(I) \cup A(L))$.

Since $v_i v_j \in A(H) \setminus (A(I) \cup A(L))$, exactly one of v_i and v_j has a loop. Assume that v_j has a loop. The case for which v_i has a loop can be treated in a similar manner.

Then we have one of the following options: (a) $i < l < j \leq k$, (b) $i < l < k < j$, (c) $l < i < k \leq j$, (d) $l < i < j < k$. However, if (c) is the case, $v_l \to v_i$ and $v_i \to v_k$, which is a contradiction since v_k, v_l belong to the same component of L^{sym} and v_i has to either dominate or be dominated by each component of L^{sym}. With a similar argument, case (d) is impossible.

Let (a) be the case. Note that v_k and v_l belong to the same component of L^{sym}. By the property of the ordering (see the proof of Lemma 3.11), v_j belongs to the same component of L^{sym} with v_k and v_l. Since the ordering of the vertices in this component satisfies the condition in Lemma 2.9, $v_l v_k \in A(L^{sym})$ and $l < j \leq k$ imply that $v_k v_j = v_{\max\{i,k\}} v_{\max\{j,l\}} \in A(L^{sym})$. By Lemma 3.11, $v_i v_l = v_{\min\{i,k\}} v_{\min\{j,l\}} \in A(H)$.

Let (b) be the case. By Lemma 3.11, both the minimum and the maximum of the two arcs are in $A(H)$. ◇

Theorem 3.13 *Let H be a semicomplete digraph with possible loops. If one of the followings holds, then MinHOM(H) is polynomial time solvable.*

(a) The digraph $H = \vec{C}_k$ for k =2 or 3.

(b) The digraph L does not contains either R or $\vec{C}_3^*$ as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph; I is a transitive tournament; H does not contain either W, R' or $\vec{C}_3$ with at least one loop as an induced subdigraph.

Proof: Consider the following cases.

Case 1: The condition (a) holds. Then, there is a polynomial time algorithm for MinHOM(H). We give the algorithm for the sake of completeness. We consider $H = \vec{C}_k$ with an arbitrary integer $k \geq 2$. We assume that the input digraph D is connected since otherwise, the algorithm can be applied to each component of D and we can sum up the costs of homomorphisms of each component to H.

Choose a vertex x of D, and assign it color 1. For any vertex y with color i, we assign all the in-neighbors of y color $i-1$ and all the out-neighbors of y color $i+1$, where the operation is taken modulo k. It is easy to see that no vertex of D is assigned a pair of conflicting colors if and only if D has a $\vec{C}_k$-coloring. Furthermore, cyclicly permutating the colors of $V(D)$ does not affect the existence of a homomorphism of D to H. Hence, we can assign x color $2, \ldots, k$, modify the assignment of other vertices of D accordingly, and compute the cost of homomorphism respectively. We finally accept an assignment which leads to the minimum cost.

Case 2: The condition (b) holds. Then by Lemma 3.12 and Theorem 2.8, MinHOM(H) is polynomial time solvable. ◇

Corollary 3.14 *Let H be a semicomplete digraph w.p.l. Then MinHOM(H) is polynomial time solvable if $H = \vec{C}_k$ for k =2 or 3, or H has a Min-Max ordering. Otherwise, MinHOM(H) is NP-hard.*

3.3 Proving that (ii-a) and (ii-b) of Theorem 3.2 are equivalent

In this subsection, we will prove that (ii-a) and (ii-b) in Theorem 3.2 are equivalent.

It follows from the proof of Lemma 3.11 that the condition (ii-a) implies (ii-b). Indeed, from the construction of the ordering in the proof of Lemma 3.11, H is a composition digraph, i.e., $H = TT_{p+l}[S_1, S_2, \ldots, S_{p+l}]$ where S_i for each $i = 1, \ldots, p+l$ is one of the two types: (a) a single vertex without a loop, (b) a reflexive semicomplete digraph which does not contain R as an induced subdigraph, and for which $U(S_i^{sym})$ is a connected proper interval graph. Here, p is the number of vertices in $V(I)$ and l is the number of components (possibly trivial) of L^{sym}.

Lemma 3.16 given below shows that the converse is also true, accomplishing the equivalence of (ii-a) and (ii-b) in Theorem 3.2.

For further reference, we give a well-known theorem that characterizes proper interval graphs in terms of forbidden subgraphs. We will start with some definitions. A graph G is called *a claw* if $V(G) = \{x_1, x_2, x_3, y\}$ and $E(G) = \{x_1y, x_2y, x_3y\}$. A graph G with $V(G) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ is called *a net* if $E(G) = \{x_1x_2, x_2x_3, x_3x_1, y_1x_1, y_2x_2, y_3x_3\}$, and *a tent* if $E(G) = \{x_1x_2, x_2x_3, x_3x_1, y_1x_2, y_1x_3, y_2x_1, y_2x_3, y_3x_1, y_3x_2\}$.

Theorem 3.15 *[16] A graph G is a proper interval graph if and only if it does not contain* *a cycle of length at least four, a claw, a net, or a tent as an induced subgraph.*

Lemma 3.16 *Let $H = TT_k[S_1, S_2, \ldots, S_k]$ where S_i for each $i = 1, \ldots, k$ is either a single vertex without a loop, or a reflexive semicomplete digraph which does not contain R as an induced subdigraph and for which $U(S_i^{sym})$ is a connected proper interval graph. Then, H is a semicomplete digraph w.p.l. such that L does not contain either R or $\vec{C}_3^*$ as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph, I is a transitive tournament and H does not contain either W, R' or $\vec{C}_3$ with at least one loop as an induced subdigraph.*

Proof: Clearly, H is a semicomplete digraph w.p.l. and I is a transitive tournament. Furthermore, the absence of W, R' or $\vec{C}_3$ with one or two loops in H follows from the transitive tournament structure of H.

Therefore, it remains to show that L does not contain $\vec{C}_3^*$ as an induced subdigraph. To the contrary, suppose that there are vertices $u, v, w \in V(L)$ such that $H[\{u, v, w\}] \cong \vec{C}_3^*$ ($u \mapsto v \mapsto w \mapsto u$). Then u, v and w must belong to the same component S_i. Since S_i^{sym} is connected, there exist paths between any pair of vertices in $\{u, v, w\}$. Define $\mu(u, v, w) = \min\{\text{dist}(u, v), \text{dist}(u, w), \text{dist}(v, w)\}$, where $\text{dist}(x, y)$ is the length of a shortest path between x and y in S_i^{sym}.

Choose a triple u, v, w in S_i such that $H[\{u, v, w\}] \cong \vec{C}_3^*$ ($u \mapsto v \mapsto w \mapsto u$) and $\mu(u, v, w)$ is minimal. Assume that $\text{dist}(u, v) = \mu(u, v, w)$. Consider a shortest a path $P = u(= u_0), u_1, \ldots, u_p(= v)$ between u and v in S_i^{sym}. Observe that $\text{dist}(u, v) \geq 2$. Let a_v (a_w) be an arc between v and u_1 (between w and u_1). If both a_v and a_w are symmetric, then u, v, w and u_1 form a claw in S_i^{sym}, which is impossible by Theorem 3.15. Hence, at most one of a_v and a_w is symmetric.

If a_v is symmetric, then a_w must be asymmetric and we have either $H[\{u, w, u_1\}] \cong R$ or $H[\{v, w, u_1\}] \cong R$, a contradiction. Similarly, if a_w is symmetric, we have either $H[\{u, v, u_1\}] \cong R$ or $H[\{w, v, u_1\}] \cong R$, also a contradiction. Hence, both a_v and a_w are asymmetric. Suppose that $u_1 \mapsto w$. Then $H[\{u, u_1, w\}] \cong \vec{C}_3^*$, a contradiction. Hence, $w \mapsto u_1$. Similarly, $u_1 \mapsto v$. Thus, u_1, v, w is a triple with $H[\{u_1, v, w\}] \cong \vec{C}_3^*$ such that $\mu(u_1, v, w) < \mu(u, v, w)$, a contradiction to the choice of u, v, w.

Thus, L does not contain $\vec{C}_3^*$ as an induced subdigraph, which completes the proof. $\diamond$

4 Further Research

We obtained a dichotomy classification for reflexive semicomplete digraphs and semicomplete digraphs w.p.l. This solves the question raised in our previous paper [9]. The obtained results imply that given a (loopless) semicomplete digraph H, for MinHOM(H) to be polynomial time solvable, H should be a very simple directed cycle or it has to be acyclic.

The problem of obtaining a dichotomy classification for semicomplete k-partite digraphs, $k \geq 2$, w.p.l. remains still open. In fact, even settling a dichotomy for k-partite tournaments seems to be not easy. Actually, for a k-partite tournament w.p.l. H, a complete dichotomy of MinHOM(H) has been obtained in [9] provided that H has a cycle. The acyclic case appears to be much harder.

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