Hamiltonian cycles avoiding prescribed arcs in tournaments

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Abstract

In [6], Thomassen conjectured that if I is a set of $k-1$ arcs in a k-strong tournament T, then $T-I$ has a Hamiltonian cycle. This conjecture was proved by Fraisse and Thomassen [3]. We prove the following stronger result. Let $T=(V,A)$ be a k-strong tournament on n vertices and let $X_1, X_2, \ldots, X_l$ be a partition of the vertex set V of T such that $|X_1| \leq |X_2| \leq \ldots \leq |X_l|$. If $k \geq \sum_{i=1}^{l-1} \lfloor |X_i|/2 \rfloor + |X_l|$, then $T - \cup_{i=1}^{l} \{xy \in A : \ x,y \in X_i\}$ has a Hamiltonian cycle. The bound on k is sharp.

1 Introduction

In [6], Thomassen conjectured that if I is a set of $k-1$ arcs in a k-strong tournament T, then $T-I$ has a Hamiltonian cycle. This conjecture was proved by Fraisse and Thomassen [3]. This result is sharp since the deletion of a set I of k arcs from a k-strong tournament may create a vertex of indegree or outdegree 0. However, the authors of [6] realized that, for some sets I, their bound was far from being the best possible (see, e.g., Section 5 in [6]).

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In this paper, we prove the following stronger result. Let $T = (V, A)$ be a k-strong tournament on n vertices and let $X_1, X_2, \dots, X_l$ be a partition of the vertex set V of T such that $|X_1| \le |X_2| \le \dots \le |X_l|$. If $k \ge \sum_{i=1}^{l-1} \lfloor |X_i|/2 \rfloor + |X_l|$, then $T - \cup_{i=1}^{l} \{xy \in A : \ x, y \in X_i\}$ has a Hamiltonian cycle. The bound on k is sharp (see Theorem 4.2).

It is easy to see that the above theorem by Fraisse and Thomassen follows from our result. Indeed, let I be a set of arcs in a tournament T, let G be the undirected graph obtained by ignoring all orientations of the arcs of $T\langle I \rangle$, the subgraph of T which has arc set I and no isolated vertices, and let $Y_1, ..., Y_m$ be the vertex sets of the connected components of G so that $|Y_1| \le ... \le |Y_m|$. By our result, T has a Hamiltonian cycle avoiding the arcs in I if T is k-strong, where $k \ge k' = \sum_{i=1}^{m-1} \lfloor |Y_i|/2 \rfloor + |Y_m|$. But $k' \le 1 + \sum_{i=1}^{m} (|Y_i| - 1) \le 1 + \sum_{i=1}^{m} e(Y_i) = 1 + |I|$, where $e(Y_i)$ is the number of edges in the component of G induced by Y_i.

A simple analysis of the last calculation shows precisely when the Fraisse-Thomassen and our theorems provide the same value of strong connectivity of T – namely, when I consists of one tree, plus maybe some independent arcs. In all other cases our result gives a better bound. In particular, if $T\langle I \rangle$ is a union of (vertex) disjoint subtournaments of T of order $n_1, ..., n_m$ ($3 \le n_1 \le ... \le n_m$), then, to guarantee that $T - I$ has a Hamiltonian cycle, we need T to be $(\sum_{i=1}^{m} \binom{n_i}{2} + 1)$-strong by the Fraisse-Thomassen theorem and to be $(n_m + \sum_{i=1}^{m-1} \lfloor n_i/2 \rfloor)$-strong by our result.

Our proof is based on Hoffmann's theorem on circulations in networks [4] and a theorem by the third author on minimal spanning 1-diregular subgraphs in semicomplete multipartite digraphs [7]. The proof of the Fraisse-Thomassen theorem was also based on a non-trivial result, namely one from [5].

2 Terminology and notation

We shall assume that the reader is familiar with the standard terminology on graphs and digraphs and refer the reader to [1].

By a *cycle* and a *path* in a directed graph we mean a directed simple cycle and path, respectively. Let D be a digraph. $V(D)$ ($A(D)$) denotes the vertex (arc) set of D. Two cycles Q and R (or paths) are *disjoint* if $V(Q) \cap V(R) = \emptyset$. A collection of vertex disjoint cycles of D is called a *cycle subgraph* of D. A cycle subgraph F of D consisting of disjoint cycles $C_1, ..., C_t$ will be denoted $F = C_1 \cup ... \cup C_t$. A digraph D is *strong* if there exists a path from x to y and a path from y to x for every choice of distinct vertices x, y of D. A digraph D is $k-$*strong* ($k > 0$) if $D - X$ is strong for any subset X of the vertices of D with $|X| < k$. Let G and H be subgraphs of D and let $P = v_1v_2...v_{k-1}v_k$ be a path in D. P is called a (G, H)-*path* if $v_1 \in V(G)$, $v_k \in V(H)$ and $\{v_2, ..., v_{k-1}\} \cap (V(G) \cup V(H)) = \emptyset$.

Let $C = x_1x_2...x_cx_1$ be a cycle in D. Then we shall usually denote x_{i+1} by x_i^+ and x_{i-1} by x_i^-, where $i = 1, 2, ..., c$ and all subscripts are taken modulo c. For a set X of vertices in C, we denote $X^+ = \{x^+ : \ x \in X\}$ and $X^- = \{x^- : \ x \in X\}$. When we use such notation, the digraph D considered has a spanning cycle subgraph F, thus every vertex x of D has a unique predecessor x^- and a unique succesor x^+ on the unique cycle in F containing x. The subpath of a cycle C from a vertex v to a vertex w will be denoted by $C[v, w]$.

If D has an arc $xy \in A(D)$, then we sometimes use the notation $x{\rightarrow}y$ and say that x *dominates* y and y *is dominated by* x. For disjoint sets X and Y of vertices in D, we say that X *strongly dominates* Y, and use the notation $X{\Rightarrow}Y$, if there is no arc from Y to X. For arbitrary sets X and Y of vertices in D, $(X, Y)_D = \{xy \in A(D) : \ x \in X, \ y \in Y\}$. In particular, if $|X| = 1$, then $(X, X)_D$ is empty.

A digraph H is called *semicomplete* if every two distinct vertices of H are adjacent. A semicomplete digraph without cycles of length two is a *tournament*. Let $X_1, X_2, \ldots, X_p$ ($p \geq 2$) be a partition of the vertex set of a semicomplete digraph H. Then the digraph $D = H - \cup_{i=1}^{p}(X_i, X_i)_H$ is called *semicomplete p-partite* or *semicomplete multipartite* (when the value of p is not important). We call $X_1, X_2, ..., X_p$ the *colour classes* of D.

3 Preliminaries

In this section we describe some results which will be important tools for the proof of our main result (Theorem 4.1). The following result is a very special case of a theorem proved by the third author in [7].

Theorem 3.1 *Let D be a semicomplete p-partite digraph with colour classes $X_1, X_2, ..., X_p$ and let F be a spanning cycle subgraph of D with the minimum* *possible number of cycles t. Then either $t = 1$, or the cycles of F can be labeled C_1,..., C_t such that the following holds: There is a pair of indices $q, q' \in \{1, 2, ..., p\}$ (q and q' may be equal), such that every arc to C_1 from the outside is in $(X_q^-, X_q^+)_D$ and every arc from C_t to outside is in $(X_{q'}^-, X_{q'}^+)_D$.*

We recall the classical theorem by Hoffmann, characterizing the existence of a feasible circulation in a network with upper and lower bounds on the arcs. Below we use the following notation. If X is a subset of the vertex set of a digraph D, then we denote by $\bar{X}$ the set $V(D) \setminus X$. If r is a function on the arc set of D, then $r(X, \bar{X}) = \sum\{r(uv) : \ uv \in (X, \bar{X})\}$. Let N be a network. A flow f in N is called an *integer* flow if $f(a)$ is integer for every arc a in N. A *circulation* is a flow of value zero in N.

Theorem 3.2 *[4] Let $N = (V, A, \ell, c)$ denote a network with vertex set V, arc set A and lower (upper) bound $\ell(a)$ ($c(a)$) on every arc $a \in A$. Suppose $\ell(a), c(a)$ are non-negative integers for each $a \in A$. There exists an integer circulation f in N such that $\ell(a) \leq f(a) \leq c(a)$ for all $a \in A$ if and only if $c(\bar{X}, X) \geq \ell(X, \bar{X})$ for all proper subsets X of V.*

For a proof of this theorem, see for example [2, page 50]. We finish this section by the following simple but useful lemma. The proof is left for the reader.

Lemma 3.3 *Let T be a tournament, and let $Y_1, Y_2, \ldots, Y_s$ ($s \geq 1$) be disjoint sets of vertices in T and let x and y be arbitrary distinct vertices in $V(T) - (Y_1 \cup Y_2 \cup \ldots \cup Y_s)$. If there exist k disjoint (x, y)-paths in T, then there exist at least $k - \sum_{i=1}^{s} \lfloor |Y_i|/2 \rfloor$ disjoint (x, y)-paths in $T - \cup_{i=1}^{s}(Y_i, Y_i)$.*

4 Results

Theorem 4.1 *Let $T = (V, A)$ be a k-strong tournament on n vertices, and let $X_1, X_2, \ldots, X_l$ ($l \geq 1$) be a partition of V ($V = \cup_{i=1}^{l} X_i, X_i \cap X_j = \emptyset$ for every $i \neq j$). Set $D = T - \cup_{i=1}^{l}(X_i, X_i)_T$, $D_i = D \cup (X_i, X_i)_T$, $D_{i,j} = D \cup (X_i, X_i)_T \cup (X_j, X_j)_T$ ($1 \leq i, j \leq l$). If $1 \leq |X_1| \leq |X_2| \leq \ldots \leq |X_l| \leq n/2$ and $k \geq |X_l| + \sum_{i=1}^{l-1} \lfloor |X_i|/2 \rfloor$, then the following hold.*

(a) *If $x \in X_i$ and $y \in X_j$ ($1 \leq i \neq j \leq l$), then there are $\lfloor |X_i|/2 \rfloor + \lfloor |X_j|/2 \rfloor + \lceil |X_l|/2 \rceil$ disjoint (x, y)-paths in $D_{i,j}$.*

(b) *If $x, y \in X_i$ ($x \neq y$), then there are $|X_i|$ disjoint (x,y)-paths in D_i. Furthermore there is an (x,y)-path in D.*

(c) *D is strong.*

(d) *D contains a spanning cycle subgraph.*

(e) *D is Hamiltonian.*

Proof:

Claim (a) easily follows from Lemma 3.3.

Proof of Claim (b): From Lemma 3.3 we easily get that there are $|X_i|$ disjoint (x,y)-paths in D_i. By deleting all arcs in $(X_i, X_i)_T$, we can destroy at most $|X_i| - 1$ of these paths, since at most $|X_i| - 2$ paths can contain vertices from $X_i - \{x, y\}$, and only one path can be the arc xy. Thus there is an (x,y)-path in D.

Proof of Claim (c): Let x and y be arbitrary distinct vertices of D. If $\{x, y\} \subseteq X_i$, then there is an (x,y)-path in D, because of Claim (b). Therefore we may now assume that $x \in X_i$ and $y \in X_j$, where $i \neq j$. Let W be the maximal set of vertices such that for all $w \in W$, there is an (x,w)-path in D.

Assume that $W \cap X_j \neq \emptyset$ and let $w \in W \cap X_j$. From (b) there is a (w,y)-path in D. Now we see that there is a (x,y)-path in D.

Assume that $W \cap X_j = \emptyset$. Therefore, there is no (x,y)-path in D_j, and at most $|X_i| - 1$ disjoint (x,y)-paths in $D_{i,j}$, since each (x,y)-path in $D_{i,j}$ must include a vertex from $X_i - x$. This is a contradiction against (a), since $\lfloor |X_i|/2 \rfloor + \lceil |X_l|/2 \rceil \geq |X_i| > |X_i| - 1$.

Hence we have proved that there exists a (x,y)-path in D for an arbitrary choice of distinct x and y, which means that D is strong.

Proof of Claim (d): Let $D' = (V' \cup V'', A(D'))$ be the digraph obtained from D by replacing each vertex $v \in V$ by two vertices v' and v'' joined by an arc from v' to v''. For each original arc $uv \in A$, D' contains the arc $u''v'$.

Let $N = (V' \cup V'', A(N), \ell, c)$ ($A(N) = A(D')$) be the network obtained from D' by specifying the following lower and upper bounds for the arcs. Every arc of the kind $u''v'$ (corresponding to an original arc in D) has lower bound zero and an upper bound ∞. Every arc $x'x''$ where $x \in V$ has lower and upper bounds equal to one.

For a subset B of V, B' and B'' will stand for the sets $\{v' : \ v \in B\}$ and $\{v'' : \ v \in B\}$, respectively.

It is easy to see that every feasible integer circulation in N corresponds to a spanning cycle subgraph in D and vice versa. Hence, by Theorem 3.2, it suffices to prove that for every proper subset U of $V' \cup V''$ we have $c(\bar{U}, U) \geq \ell(U, \bar{U})$.

Assume that $c(\bar{U}, U) < \ell(U, \bar{U})$, where U is a proper subset of $V' \cup V''$. The vertex set V of D can be partitioned into the following sets: $Y = \{y \in V : \ y' \in U, \ y'' \in \bar{U}\}$, $Z = \{z \in V : \ z'' \in U, \ z' \in \bar{U}\}$, $R_1 = \{v \in V : \{v', v''\} \subseteq U\}$ and $R_2 = \{v \in V : \ \{v', v''\} \subseteq \bar{U}\}$. We have $|Z| < |Y|$ since $|Z| \leq c(\bar{U}, U)$ and $\ell(U, \bar{U}) = |Y|$.

Observe that an arc $uv \in A(D)$ between two vertices in $Y \subseteq V(D)$ will correspond to an arc $u''v'$ in N with $u'' \in \bar{U}$ and $v' \in U$, thus contributing ∞ to $c(\bar{U}, U)$. Therefore there is an $i \in \{1, 2, \ldots, l\}$ such that $Y \subseteq X_i$. If D has an arc yr_1 from Y to R_1, then $y''r_1' \in A(N)$ contributes ∞ to $c(\bar{U}, U)$. Hence, $R_1 \Rightarrow Y$. Analogously, $(R_1 \cup Y) \Rightarrow R_2$. Since $|Z| < |Y| \leq |X_i| \leq n/2$ and $Y \subseteq X_i$, we obtain that either $W_1 = R_1 - X_i$ or $W_2 = R_2 - X_i$ is non-empty. Since $X_1, ..., X_l$ form a partition of $V(D)$, there is j, $j \neq i$, so that $(W_1 \cup W_2) \cap X_j \neq \emptyset$.

Assume that $|W_1 \cap X_j| \geq |W_2 \cap X_j|$ and choose a vertex w_1 in $W_1 \cap X_j$ and vertex $y \in Y$. Set $S = (W_2 \cap X_j) \cup Z \cup (X_i - Y)$. By (a) there exist at least $|X_i| + \lfloor |X_j|/2 \rfloor$ disjoint (y, w_1)-paths in $D_{i,j}$. However, $D_{i,j} - S$ has no (y, w_1)-paths and $|S| \leq \lfloor |X_j|/2 \rfloor + |Z| + |X_i| - |Y| < \lfloor |X_j|/2 \rfloor + |X_i|$; a contradiction.

If $|W_1 \cap X_j| < |W_2 \cap X_j|$, then choose a vertex w_2 in $W_2 \cap X_j$ and vertex $y \in Y$. Set $S = (W_1 \cap X_j) \cup Z \cup (X_i - Y)$. Now $D_{i,j} - S$ has no (w_2, y)-path and we obtain a contradiction as above.

Proof of (e): Assume that D is not Hamiltonian.

We first observe that D is a semicomplete multipartite digraph with colour classes $X_1, X_2, \ldots, X_l$. By (c) the digraph D is strong. By (d) D contains a spanning cycle subgraph. Let F be a spanning cycle subgraph of D with the minimal possible number of cycles t. Assume that $t \geq 2$. Let C_1,..., C_t be a labeling of the cycles of F determined in Theorem 3.1. By Theorem 3.1, there is a pair of indices $q, q' \in \{1, 2, ..., p\}$ such that the conclusion of the theorem holds. Let us fix the labeling and pair of indices above.

If $|V(C_1)| > n/2$, we reverse all the arcs in D and relabel the cycles in the spanning cycle subgraph corresponding to F. Hence, we may assume that $|V(C_1)| \leq n/2$. Moreover assume that $|V(C_1)|$ has the minimal possible value. Set $W = V(C_2) \cup \ldots \cup V(C_t)$. For all $i = 1, 2, \ldots, l$ set $z_i = |V(C_1) \cap X_i|$.

Assume that $z_q = |V(C_1)|/2$. There are $|V(C_1)|(|V(C_1)| - 1)/2$ arcs in $(V(C_1), V(C_1))_T$, and, by Theorem 3.1, there are at most $z_q(|X_q| - z_q)$ arcs in $(W, V(C_1))_D$. Let $i \in \{1, ..., l\}$. Since $z_i/|V(C_1)| \leq 1/2$, we have $|(X_i \cap W, X_i \cap V(C_1))_T| \leq z_i(|X_i| - z_i) \leq |V(C_1)|\lfloor |X_i|/2 \rfloor$. Now we obtain the following:

$$\begin{aligned} |(V, V(C_1))_T| &\leq |V(C_1)|(|V(C_1)| - 1)/2 + z_q(|X_q| - z_q) + \textstyle\sum_{i=1}^{l} z_i(|X_i| - z_i) \\ &\leq |V(C_1)|(|V(C_1)| - 1)/2 + 2z_q(|X_q| - z_q) + \textstyle\sum_{\substack{i=1 \\ i\neq q}}^{l} |V(C_1)|\lfloor |X_i|/2 \rfloor \\ &= |V(C_1)|\Big[|V(C_1)|/2 - 1/2 + |X_q| - |V(C_1)|/2 + \textstyle\sum_{\substack{i=1 \\ i\neq q}}^{l} \lfloor |X_i|/2 \rfloor\Big] \\ &< |V(C_1)|\Big[|X_q| + \textstyle\sum_{\substack{i=1 \\ i\neq q}}^{l} \lfloor |X_i|/2 \rfloor\Big]. \end{aligned}$$

This implies that there is a vertex w in $V(C_1)$ such that

$$|(V, w)_T| < |X_q| + \sum_{i=1,\, i\neq q}^{l} \lfloor |X_i|/2 \rfloor \leq k,$$

which is a contradiction against the fact that T is k-strong. Therefore we have shown that $|V(C_1) \cap X_q| < |V(C_1)|/2$.

Let

$$S = \{s \in V(C_1) : \text{ there exists an arc from } W \text{ to } s^+ \text{ in } D\}.$$

Note that, by Theorem 3.1, $S \subseteq X_q \cap V(C_1)$. Since $|V(C_1) \cap X_q| < |V(C_1)|/2$, the set $R = V(C_1) - (X_q \cup X_q^+)$ is not empty. We prove that $R \Rightarrow S$ in D. Assume that there exist $s \in S$ and $r \in R$ so that $sr \in A(D)$. There is vertex $w \in V(C_i)$ (for some i, $2 \leq i \leq l$) so that $ws^+ \in A(D)$. Since $w^+ \in X_q$ and $r^- \notin X_q$, the arc $r^-w^+ \in A(D)$ ($w^+r^- \notin A(D)$ because $w^{++} \notin X_q$). Replace the cycles C_1 and C_i by the cycles $C_1[r, s]r$ and $C_i[w^+, w]C_1[s^+, r^-]w^+$ in F. The new spanning cycle subgraph F' has t cycles as well. However, the first cycle of F' contains less vertices than C_1 does; a contradiction.

Let X_j be a colour class of D so that $X_j \cap R$ is not empty. Let r be a vertex in $X_j \cap R$ and let w_q be a vertex in $X_q \cap W$. Consider two cases.

Case 1: S is a proper subset of $X_q \cap V(C_1)$.

Let $x \in (X_q \cap C_1) - S$. For all colour classes, $X_i \neq X_q$ we have $|X_i \cap V(C_1)| > |X_i \cap W|$ because of the following. Assume that there is a colour class, $X_i \neq X_q$, with $|X_i \cap V(C_1)| \leq |X_i \cap W|$. Let $w \in X_i \cap W$ be arbitrary. Claim (a) implies that there exist $\lfloor |X_i|/2 \rfloor + |X_q|$ disjoint (w, x)-paths in $D_{i,q}$. However, $B = [(W \cap X_q) \cup S^+] \cup (X_i \cap V(C_1))$ separates x from w in $D_{i,q}$ and $|B| < |X_q| + \lfloor |X_i|/2 \rfloor$; a contradiction.

Since $|V(C_1)| \leq |V(D)|/2$, we obtain $|X_q \cap V(C_1)| \leq |X_q \cap W|$. Claim (a) implies that there exist $\lfloor |X_j|/2 \rfloor + |X_q|$ disjoint (w_q, r)-paths in $D_{j,q}$. However, $B = [(V(C_1) \cap X_q) \cup S^+] \cup (X_j \cap W)$ separates r from w_q in $D_{j,q}$ and $|B| < |X_q| + \lfloor |X_j|/2 \rfloor$; a contradiction.

Case 2: $S = X_q \cap V(C_1)$.

Subcase 2a: $|X_j \cap V(C_1)| \geq |X_j \cap W|$ or $|S| \leq |X_q \cap W|$.

Set $B = S^+ \cup (X_j \cap W)$. The digraph $D_{j,q} - B$ has no (w_q, r)-paths since $R \Rightarrow S$ in D. If $|X_j \cap V(C_1)| \geq |X_j \cap W|$, then $|B| < \lfloor |X_j|/2 \rfloor + |X_q|$ since $X_q \cap W$ is not empty. If $|S| \leq |X_q \cap W|$, then $|B| < \lfloor |X_q|/2 \rfloor + |X_j|$ since $X_j \cap V(C_1)$ is not empty. Each of the last two bounds for $|B|$ contradicts (a).

Subcase 2b: $|X_j \cap V(C_1)| < |X_j \cap W|$ and $|S| > |X_q \cap W|$.

If $X_j \cap W \subseteq (X_q \cap W)^-$, then set $B = S^+ \cup (X_j \cap W)$. The digraph $D_{j,q} - B$ has no (w_q, r)-path. However, $|B| \leq |X_q| < \lfloor |X_j|/2 \rfloor + |X_q|$; a contradiction to (a).

If $(X_j \cap W) - (X_q \cap W)^- \neq \emptyset$, then set $B = (X_q \cap W) \cup (X_q \cap W)^- \cup (X_j \cap V(C_1))$. The digraph $D_{j,q} - B$ has no (w, x)-path, where $x \in V(C_1) \cap X_q$ and $w \in (X_j \cap W) - (X_q \cap W)^-$. However, $|B| < \lfloor |X_j|/2 \rfloor + |X_q|$; a contradiction to (a).

Hence, we have got a contradiction in both cases which implies that D is Hamiltonian. $\square$.

The bound for k in Theorem 4.1 is sharp because of the following theorem.

Theorem 4.2 *Let $2 \leq r_1 \leq r_2 \leq \ldots \leq r_l$ be arbitrary integers, then there exists a tournament T and a collection $X_1, X_2, ..., X_l$ of disjoint sets of vertices in T such that*

(i) *T is $(r_l - 1 + \sum_{i=1}^{l-1} \lfloor r_i/2 \rfloor)$-strong;*

(ii) *$|X_i| = r_i$ for $i = 1, 2, \ldots, l$;*

(iii) *$D = T - \cup_{i=1}^{l} (X_i, X_i)_T$ is not Hamiltonian.*

Proof: Let $k = r_l + \sum_{i=1}^{l-1} \lfloor r_i/2 \rfloor$. We construct a tournament T with the properties (i)-(iii): $V(T) = X_1 \cup X_2 \cup \ldots \cup X_l \cup Y_1' \cup Y_2' \cup Y_1'' \cup Y_2'' \cup Z$, where all the sets in the union are mutually disjoint. Let $|Y_1'| = |Y_2'| = |Y_1''| = |Y_2''| = k$, $|Z| = r_l - 1$, $X_j = \{x_{j1}, ..., x_{j,r_j}\}$, $X_j' = \{x_{j1}, ..., x_{j,\lfloor r_i/2 \rfloor}\}$ and $X_j'' = \{x_{j,\lfloor r_i/2 \rfloor + 1}, ..., x_{j,r_j}\}$ for every $j \in \{1, ..., l-1\}$. Set $S' = (\cup_{j=1}^{l-1} X_j') \cup Y_1' \cup Y_2'$, $S'' = (\cup_{j=1}^{l-1} X_j'') \cup Y_1'' \cup Y_2''$. The arc set of T is defined as follows. Let $S' \Rightarrow X_l \Rightarrow S'' \Rightarrow Z \Rightarrow S'$. If $s' \in S'$ and $s'' \in S''$, then $s' \rightarrow s''$ unless there exists $j \in \{1, ..., l-1\}$ so that both s' and s'' are from X_j in which case $s'' \rightarrow s'$. For every $i = \{', ''\}$ and $j \in \{1, ..., l-1\}$, $X_j^i \Rightarrow Y_1^i \Rightarrow Y_2^i \Rightarrow X_j^i$. The direction of the arcs between the vertices non-adjacent so far can be chosen arbitrary.

To see that T is $(k-1)$-strong, note that the deletion of any $k-2$ vertices leaves at least one vertex y_i^j in each of the sets Y_i^j, and either (a) a vertex z in Z, or (b) an edge $s'' \rightarrow s'$ from some X_j'' to the corresponding X_j'. In case (a), there is a cycle $zy_1'y_2'y_1''y_2''z$ remaining, and in case (b) we have the cycle $s'y_1'y_2'y_1''y_2''s''s'$. In either case, every other vertex sends and receives at least one edge to/from the cycle, so the remaining digraph is strong.

Assume that $D = T - \cup_{i=1}^{l} (X_i, X_i)_T$ is Hamiltonian. In a Hamiltonian cycle of D, after every visit to X_l, the cycle must pass through Z before returning to X_l since $A(D - Z) \subseteq (S', S'')_T \cup (S', X_l)_T \cup (X_l, S'')_T$. However, there are more vertices of X_l than of Z, thus there is no Hamiltonian cycle. $\square$.

5 Conclusions and open problems

In fact, this paper is concerned with aspects of the following general question. Which sets B of edges of the complete graph K_n have the property that every k-strong orientation of K_n induces a Hamiltonian digraph on $K_n - B$? The

Fraisse-Thomassen theorem says that this is the case whenever A contains at most $k-1$ edges. Here, it has been shown that a union of disjoint cliques of sizes $r_1, \ldots, r_l$ has the property, whenever $\sum_{i=1}^{l} \lfloor r_i/2 \rfloor + \max_{1 \le i \le l}\{\lceil r_i/2 \rceil\} \le k$. This is the best possible result for unions of cliques. Also, it implies the Fraisse-Thomassen theorem.

It seems natural to investigate bounds for k in different cases of the set B. In particular, what are sharp bounds for k when B is a spanning forest of K_n consisting of m disjoint paths containing r_1, ..., r_m vertices, respectively ? The same question can be asked if we replace "paths" by "stars" or by "cycles" (in the last case "spanning forest" should also be replaced by "spanning cycle subgraph").

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