# Properly coloured Hamiltonian paths in edge-coloured complete graphs

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**Abstract**

We consider edge-coloured complete graphs. A path or cycle $Q$ is called properly coloured (PC) if any two adjacent edges of $Q$ differ in colour. Our note is inspired by the following conjecture by B. Bollobás and P. Erdős (1976) : if $G$ is an edge-coloured complete graph on $n$ vertices in which the maximum monochromatic degree of every vertex is less than $\lfloor n/2 \rfloor$, then $G$ contains a PC Hamiltonian cycle. We prove that if an edge-coloured complete graph contains a PC 2-factor then it has a PC Hamiltonian path. R. Häggkvist (1996) announced that every edge-coloured complete graph satisfying Bollobás-Erdős condition contains a PC 2-factor. These two results imply that every edge-coloured complete graph satisfying Bollobás-Erdős condition has a PC Hamiltonian path.

## 1 Introduction

Properly coloured Hamiltonian paths and cycles in edge-coloured graphs have applications in genetics (cf. [7, 8, 9]) and social sciences (cf. [6]) besides a number of applications in graph theory and algorithms. A path or cycle $Q$ is called *properly coloured* (abbreviated to *PC*) if any two adjacent edges of $Q$ differ in colour.

In our note, we consider edge-coloured complete graphs. We use the notation $K_n^c$ to denote a complete graph on $n$ vertices, each edge of which is coloured by a colour from the set $\{1, 2, \ldots, c\}$. Our note is inspired by the following conjecture due to B.

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Bollobás and P. Erdős [4]: if $\Delta(K_n^c) < \lfloor n/2 \rfloor$ then $K_n^c$ contains a PC Hamiltonian cycle. Here $\Delta(K_n^c)$ is the maximum number of edges of the same colour adjacent to a vertex of $K_n^c$.

B. Bollobás and P. Erdős [4] managed to prove that $\Delta(K_n^c) < n/69$ implies the existence of a properly coloured Hamiltonian cycle in $K_n^c$. This result was improved by C.C. Chen and D.E. Daykin [5] to $\Delta(K_n^c) < n/17$ and by J. Shearer [11] to $\Delta(K_n^c) < n/7$. Recently, N. Alon and G. Gutin [1] proved that for every $\epsilon > 0$ there exists an $n_0(\epsilon)$ so that for each $n > n_0(\epsilon)$, $K_n^c$ satisfying $\Delta(K_n^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)n$ has a PC Hamiltonian cycle.

In our note the following result is shown:

**Theorem 1.1** *If $K_n^c$ contains a properly coloured 2-factor, then it has a properly coloured Hamiltonian path.*

Another sufficient condition for $K_n^c$ to contain a PC Hamiltonian path was found by O. Barr [3]: every $K_n^c$ without monochromatic triangles has a PC Hamiltonian path. The following necessary and sufficient conditions for the existence of a PC Hamiltonian path in $K_n^c$ were conjectured in the survey paper [2].

**Conjecture 1.2** *A $K_n^c$ has a PC Hamiltonian path if and only if $K_n^c$ contains a collection $F$ consisting of a PC path $P$ and a number of cycles $C_1, ..., C_t$ $(t \geq 0)$, each PC, such that the members of $F$ are pairwise vertex disjoint and $V(P \cup C_1 \cup ... \cup C_t) = V(K_n^c)$.*

Theorem 1.1 provides some support to the conjecture. In [2], the conjecture was verified for the case of two colours ($c = 2$). The proof in [2] is indirect and uses the corresponding result on Hamiltonian directed paths in bipartite tournaments. For the sake of completeness, we give a short direct proof of the case $c = 2$ of Conjecture 1.2 in the next section.

R. Häggkvist [10] announced a non-trivial proof of the fact that every edge-coloured complete graph satisfying Bollobás-Erdős condition contains a PC 2-factor. Theorem 1.1 and Häggkvist's result imply that every edge-coloured complete graph satisfying Bollobás-Erdős condition has a PC Hamiltonian path.

For a set of vertices $W$ of $K_n^c$ and a vertex $x$ not in $W$, we denote by $xW$ the set of edges between $x$ and $W$, i.e. $xW = \{xy : y \in W\}$; if all the edges of $xW$ are of the same colour $k$, then $\chi(xW)$ denotes the colour $k$.

## 2 Proofs

*Proof of Theorem 1.1:*

Let $C_1, C_2, ..., C_t$ be the cycles of a PC 2-factor $F$ in $K_n^c$. Let $F$ be chosen so that, among all PC 2-factors of $K_n^c$, the number of cycles $t$ is minimum. We say that $C_i$ *dominates* $C_j$ $(i \neq j)$ if, for every edge $xy$ of $C_i$, there exists an edge between $x$ and $C_j$ and an edge between $y$ and $C_j$ whose colours differ from the colour of $xy$. Construct a digraph $D$ as follows. The vertices of $D$ are $1, 2, ..., t$ and an arc $(i, j)$ is in $D$ $(1 \leq i \neq j \leq t)$ if and only if $C_i$ dominates $C_j$.

First we show that $D$ is semicomplete, i.e. every pair of vertices of $D$ are adjacent. Suppose this is not so, i.e. there exist vertices $i$ and $j$ which are not adjacent. This means that neither $C_i$ dominates $C_j$ nor $C_j$ dominates $C_i$. Thus $C_i$ has an edge $xy$ such that $\chi(xV(C_j)) = \chi(xy)$ and $C_j$ has an edge $uv$ such that $\chi(uV(C_i)) = \chi(uv)$. It follows that $\chi(xy) = \chi(xu) = \chi(uv) = \chi(xv) = \chi(uy)$. Therefore, we can merge the two cycles to obtain a new properly coloured one as follows: delete $xy$ and $uv$, and append $xv$ and $yu$. However, this is a contradiction to $t$ being minimum. Thus, $D$ is indeed semicomplete.

Since $D$ is semicomplete, it follows from the well-known Redei theorem that $D$ has a Hamiltonian directed path: $(i_1, i_2, ..., i_t)$. Without loss of generality we may assume that $i_k = k$ for every $k = 1, 2, ..., t$. In other words, $C_i$ dominates $C_{i+1}$ for every $1 \leq i \leq t-1$.

Let $C_i = z_1^i z_2^i ... z_{m_i}^i z_1^i$ $(i = 1, 2, ..., t)$. As $C_1$ dominates $C_2$, without loss of generality, we may assume the labelings of the vertices in $C_1$ and $C_2$ are such that $\chi(z_{m_1}^1 z_1^1) \neq \chi(z_1^1 z_2^2)$. Since the edges $z_1^2 z_2^2$ and $z_2^2 z_3^2$ have different colours, without loss of generality we may assume that $\chi(z_2^2 z_3^2) \neq \chi(z_1^1 z_2^2)$. Analogously, for every $i = 1, 2, ..., t-1$, we may assume that $\chi(z_{m_i}^i z_1^i) \neq \chi(z_1^i z_2^{i+1}) \neq \chi(z_2^{i+1} z_3^{i+1})$. Now we obtain the following PC Hamiltonian path:

$$z_2^1 z_3^1 ... z_{m_1}^1 z_1^1 z_2^2 z_3^2 ... z_{m_2}^2 z_1^2 ... z_2^t z_3^t ... z_{m_t}^t z_1^t.$$

□

*Proof of Conjecture 1.2 for the case $c = 2$:*

It is easy to see that to prove our claim it suffices to show that if $K_n^2$ has a PC path $P$ and a PC cycle $C$ such that $V(P) \cap V(C) = \emptyset$ and $V(P \cup C) = V(K_n^2)$, then $K_n^2$ contains a PC Hamiltonian path.

Assume that $K_n^2$ has no PC Hamiltonian path.

Let $P = x_1x_2...x_k$ and $C = y_1y_2...y_my_1$. If there exists $i \in \{1, 2, ..., m\}$ such that $\chi(x_1x_2) \neq \chi(x_1y_i)$, then at least one of the following two Hamiltonian paths is properly coloured: $y_{i+1}y_{i+2}...y_my_1...y_ix_1x_2...x_k$; $y_{i-1}y_{i-2}...y_1y_my_{m-1}...y_ix_1...x_k$. Thus, we conclude that $\chi(x_1x_2) = \chi(x_1V(C))$. Analogously, we can prove that $\chi(x_kV(C)) = \chi(x_{k-1}x_k)$.

Suppose that we have proved that $\chi(x_{j-1}x_j) = \chi(x_{j-1}V(C))$ for some $j \in \{2, ..., k-1\}$. Then $\chi(x_jx_{j+1}) = \chi(x_jV(C))$ holds. Indeed, assume that there is $i \in \{1, ..., k\}$ such that $\chi(x_jx_{j+1}) \neq \chi(x_jy_i)$. As $c = 2$, we may assume without loss of generality that $\chi(x_{j-1}x_j) = \chi(y_{i-1}y_i) = 1$. Again, since $c = 2$, we obtain that $\chi(x_jy_i) = \chi(x_{j-1}y_{i-1}) = 1$. Thus, $x_1...x_{j-1}y_{i-1}y_{i-2}...y_1y_m...y_ix_jx_{j+1}...x_k$ is a PC Hamiltonian path in $K_n^2$; a contradiction.

Now, by induction, we conclude that $\chi(x_{k-1}x_k) = \chi(x_{k-1}V(C))$. Recall that $\chi(x_kV(C)) = \chi(x_{k-1}x_k)$. Without loss of generality, assume that $\chi(y_1y_2) = \chi(x_{k-1}x_k)$. Hence, $x_ky_2y_3...y_my_1x_{k-1}x_{k-2}...x_1$ is a PC Hamiltonian path in $K_n^2$; a contradiction. $\square$

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