Properly colored Hamilton cycles in edge colored complete graphs

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Dedicated to the memory of Paul Erdős

Abstract

It is shown that for every $\epsilon > 0$ and $n > n_0(\epsilon)$, any complete graph K on n vertices whose edges are colored so that no vertex is incident with more than $(1 - \frac{1}{\sqrt{2}} - \epsilon)n$ edges of the same color, contains a Hamilton cycle in which adjacent edges have distinct colors. Moreover, for every k between 3 and n any such K contains a cycle of length k in which adjacent edges have distinct colors.

1 Introduction

Let G^c denote a graph G whose edges are colored in an arbitrary way. In particular, K_n^c denotes an edge-colored complete graph on n vertices and $K_{m,m}^c$ denotes an edge-colored complete bipartite graph with equal partite sets of cardinality m each. For an edge-colored graph G^c, let $\Delta(G^c)$ denote the maximum number of edges of the same color incident with a vertex of G^c. A properly colored cycle in G^c, that is, a cycle in which adjacent edges have distinct colors is called an *alternating cycle*. In particular, an *alternating Hamilton cycle* is a properly colored Hamilton cycle in G^c. Bollobás and Erdős [6] proved that if $\Delta(K_n^c) < n/69$ then K_n^c contains an alternating Hamilton cycle. This was improved by Chen and Daykin [8] and Shearer [10] who proved that the same conclusion holds under the weaker assumptions $\Delta(K_n^c) \leq n/17$ and $\Delta(K_n^c) < n/7$, respectively. The authors of [6] conjectured that in fact it is enough to assume that $\Delta(K_n^c) < \lfloor n/2 \rfloor$ which, if true, would be best possible. In this note we prove the following theorem, which improves the estimate of [10], but still falls short of establishing the above mentioned conjecture.

Theorem 1.1 *For every $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon)$ so that for every $n > n_0$, every K_n^c satisfying*

$$\Delta(K_n^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)n \quad (\quad = (0.2928... - \epsilon)n \quad) \tag{1}$$

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contains an alternating Hamilton cycle.

Our proof combines probabilistic arguments with some of the known results on *directed* Hamilton cycles in digraphs. The basic idea for an even $n = 2m$ is the following (the proof for the odd case is similar). Given a K_n^c satisfying (1), split the set of vertices randomly into two disjoint subsets A and B of cardinality m each, and let a_ib_i, $(1 \leq i \leq m)$ be a random matching between the members of A and those of B. Construct a digraph $D = (V, E)$ on the set $V = \{v_1, v_2, \ldots, v_m\}$ by letting v_iv_j be a directed edge (for $i \neq j$) iff the color of a_ib_j in K_n^c differs from that of a_ib_i and that of a_jb_j. By applying some large deviation inequalities we show that for large m, with high probability every indegree and every outdegree of D exceeds $m/2$, implying, by a known result of Nash-Williams, that D contains a directed Hamilton cycle. This yields an alternating Hamilton cycle that contains all matching edges a_ib_i in K_n^c. The detailed proof appears in the next section. The final section contains some related remarks and extensions to cycles of other lengths.

For additional information on properly colored paths and cycles we refer the reader to the survey paper [3].

2 The Proof

In this section we prove Theorem 1.1. For simplicity we assume first that $n = 2m$ is even, and remark at the end of the section how to modify the argument for the case of odd n. Fix a positive ϵ, and let $K = K_n^c$ be an edge colored complete graph on $n = 2m$ vertices satisfying (1). We first prove the following simple lemma (similar results are proved in various places, see, e.g., [1]).

Lemma 2.1 *For all sufficiently large m, K contains a spanning edge colored complete bipartite graph $K_{m,m}^c$ satisfying*

$$\Delta(K_{m,m}^c) \leq (1 - \frac{1}{\sqrt{2}} - \frac{\epsilon}{2})m. \tag{2}$$

Proof. Let u_iv_i, $(1 \leq i \leq m)$ be an arbitrary perfect matching in K and choose a random partition of the set of vertices of K into two disjoint subsets A and B of cardinality m each by choosing, for each i, $1 \leq i \leq m$, randomly and independently, one element of the set $\{u_i, v_i\}$ to be a member of A and the other to be a member of B. Fix a vertex w of K and a color, say red, that appears in the edge-coloring of K. The number of neighbors a of w in A so that the edge wa is red can be written as a sum of m independent indicator random variables $x_1, \ldots, x_m$, where x_i is the number of red neighbors of w in A among u_i, v_i. Thus each x_i is either 1 with probability one (in case both edges wu_i, wv_i are red) or 0 with probability 1 (in case none of the edges wu_i, wv_i is red) or 1 with probability 1/2 (in case exactly one of these two edges is red). It follows that if the total number of red edges incident with w is r then the probability that w has more than $(r + s)/2$ red neighbors in A is equal to the probability that more than $(q + s)/2$ flips among q independent flips of a fair coin give "heads", where q is the number of nonconstant indicator random variables among the x_i's. This can be bounded by the well known inequality of Chernoff (cf., e.g., [2], Theorem A.4, page 235) by $e^{-2s^2/q} < e^{-2s^2/m}$. Since the same argument applies to the number of red neighbors of w in B, and since there are less than $8m^3$ choices for a vertex w, a color in the given coloring of K and a partite set (A or B), we conclude that the probability that there exists a vertex with more than

$$(1 - \frac{1}{\sqrt{2}} - \frac{\epsilon}{2})m$$

neighbors of the same color in either A or B is at most

$$8m^3 e^{-2\epsilon^2 m},$$

which is (much) smaller than 1 for all sufficiently large m. Therefore, there exists a choice for A and B so that the above does not occur, completing the proof. □

The next lemma is proved by applying a large deviation result for martingales.

Lemma 2.2 *Let U be a subset of $M = \{1, 2, \ldots, m-1\}$ and suppose that for each $u \in U$ there is a subset $S_u \subset M$, where $|S_u| \leq r$ for all u. Let $f : U \mapsto M$ be a random one-to-one mapping of U into M, chosen uniformly among all one-to-one mappings of U into M, and define:*

$$B(f) = |\{u \in U : f(u) \in S_u\}|.$$

Then the expectation of $B(f)$ is given by

$$E = E(B(f)) = \sum_{u \in U} \frac{|S(u)|}{m-1} \quad (\leq \frac{|U|r}{m-1},)$$

and the probability that $B(f)$ is larger satisfies the following inequality. For every $\lambda > 0$

$$Prob[B(f) - E > 4\lambda\sqrt{m-1}] < e^{-\lambda^2}.$$

Proof. For each fixed $u \in U$, the probability that $f(u) \in S_u$ is precisely $|S(u)|/(m-1)$, and the claimed expression for the expectation of $B(f)$ thus follows from linearity of expectation. To prove the second assertion in the lemma we apply a martingale inequality of Azuma (cf., e.g., [2], Chapter 7). Let $u_1, u_2, \ldots, u_l$ be all elements of U ($|U| = l \leq m-1$). Define a martingale $X_0, X_1, \ldots, X_l$ by setting

$$X_i(g) = E[B(f) : f(u_j) = g(u_j) \text{ for all } j \leq i].$$

Therefore X_0 is a constant and equals the expectation E of $B(f)$, whereas X_l is $B(f)$ itself. Moreover, if $f, f' : U \mapsto M$ differ only on k members of U then $|B(f) - B(f')| \leq k$. Therefore, using the technique in [9], pp. 33-35 or in [2], pp. 89-92 one can prove that for each $i, 0 \leq i \leq l-1$, $|X_{i+1}(g) - X_i(g)| \leq 2$. Here are the details. Consider, first, two one-to-one functions $g, g' : U \mapsto M$ that agree on $\{1, 2, \ldots, i\}$ but may differ on $i+1$. For each one-to-one $f : U \mapsto M$ that agrees with g on $\{1, 2, \ldots, i+1\}$, define a function $f' : U \mapsto M$ as follows. $f'(u_j) = g'(u_j)$ for $j \leq i+1$.

If $g'(u_{i+1}) \notin f(U)$ then $f'(u_j) = f(u_j)$ for all $j > i+1$. Otherwise, suppose $f(u_{i^*}) = g'(u_{i+1})$. In this case, define $f'(u_{i^*}) = g(u_{i+1})$ and $f'(u_j) = f(u_j)$ for all $j > i+1, j \neq i^*$. Note that $|B(f) - B(f')| \leq 2$, as f and f' differ on at most two points. Moreover, the correspondence between f and f' is a bijection between all possible one-to-one extensions of g and those of g'. Therefore $|X_{i+1}(g) - X_{i+1}(g')| \leq 2$, and as $X_i(g)$ is a weighted average of quantities of the form $X_{i+1}(g')$ for functions g' as above, it follows that $X_i(g)$ cannnot differ from any of those by more than 2, and hence $|X_i(g) - X_{i+1}(g)| \leq 2$, as claimed.
This, together with the Azuma Inequality and the method in the above references supplies the desired estimate for the probability that $B(f)$ exceeds $E + 4\lambda\sqrt{m-1}$. □

Corollary 2.3 *Let $K^c_{m,m}$ be an edge colored complete bipartite graph on the partite sets A and B, and suppose that (2) holds. Then, for all sufficiently large m, there exists a perfect matching $a_i b_i$, $1 \leq i \leq m$, in $K^c_{m,m}$ so that the following two conditions hold.*
(i) For every i the number $d^+(i)$ of edges $a_i b_j$ between a_i and B whose colors differ from those of $a_i b_i$ and of $a_j b_j$ is at least $m/2 + 1$.
(ii) For every j the number $d^-(j)$ of edges $a_i b_j$ between b_j and A whose colors differ from those of $a_i b_i$ and of $a_j b_j$ is at least $m/2 + 1$.

Proof. Let $a_i b_i$, $1 \leq i \leq m$, be a random perfect matching between A and B, chosen among all possible matchings with uniform probability. Put $r = \Delta(K^c_{m,m})$ and notice that by (2)

$$r \leq (1 - \frac{1}{\sqrt{2}} - \frac{\epsilon}{2})m.$$

Fix an i, say $i = m$, and let us estimate the probability that the condition (i) fails for i. Suppose the edge $a_m b_m$ has already been chosen for our random matching, and the rest of the matching still has to be chosen randomly. There are at most r edges $a_m b$, $(b \in B)$ having the same color as $a_m b_m$. Let U be the set of all the remaining elements B. Then $|U| \geq m - r$. For each $u \in U$, let S_u denote the set of all elements $a \in A - a_m$ so that the color of the edge au is equal to that of the edge $a_m u$. The random matching restricted to U is simply a random one-to-one function f from U to $A - a_m$. Moreover, the edge $a_m u$ will not be counted among the edges incident with a_m and having colors that differ from those of $a_m b_m$ and of the edge matched to u if and only if the edge matched to u will lie in S_u. It follows that the random variable counting the number of such edges of the form $a_m u$ behaves precisely like the random variable $B(f)$ in Lemma 2.2. By choosing say, $\lambda = \sqrt{\log(4m)}$ we conclude that the probability that $B(f)$ exceeds $|U|r/(m-1) + 4\lambda\sqrt{m-1}$ is smaller than $1/(4m)$. Therefore, with probability at least $1 - \frac{1}{4m}$

$$d^+(m) \geq |U| - |U|r/(m-1) - 4\sqrt{m}\sqrt{\log(4m)} \geq \frac{(m-r)(m-r-1)}{m-1} - 4\sqrt{m}\sqrt{\log(4m)} > m/2 + 1,$$

for all sufficiently large m, (using the fact that $r \leq (1 - \frac{1}{\sqrt{2}} - \frac{\epsilon}{2})m$.)

Since there are m choices for the vertex a_i (and similarly m choices for the vertex b_j for which the computation is similar) we conclude that with probability at least a half $d^+(i) > m/2+1$, and $d^-(j) > m/2+1$ for all i and j. In particular there exists such a matching, completing the proof of the corollary. □

To complete the proof of Theorem 1.1 we need the following result of Nash-Williams (cf., e.g., [5], page 201); for some stronger sufficient conditions for a digraph to be Hamiltonian see, e.g., [4, 7].

Theorem 2.4 (Nash-Williams) *Any directed graph on m vertices in which every indegree and every outdegree is at least $m/2$ contains a directed Hamilton cycle.*

Returning to the proof of Theorem 1.1 with $n = 2m$, and given an edge colored K_n^c satisfying (1) apply Lemma 2.1 and Corollary 2.3 to obtain a matching a_ib_i satisfying the two conditions in the corollary. Construct a digraph $D = (V, E)$ on the set of vertices $V = \{v_1, v_2, \ldots, v_m\}$ by letting v_iv_j be a directed edge (for $i \neq j$) iff the color of a_ib_j in K_n^c differs from that of a_ib_i and that of a_jb_j. By Corollary 2.3 every indegree and every outdegree of D exceeds $m/2$, implying, by Theorem 2.4, that D contains a directed Hamilton cycle $v_{\pi(1)}v_{\pi(2)}\ldots v_{\pi(m)}v_{\pi(1)}$, where $\pi = \pi(1), \pi(2), \ldots, \pi(m)$ is a permutation of $\{1, 2, \ldots, m\}$. The cycle $b_{\pi(1)}a_{\pi(1)}b_{\pi(2)}a_{\pi(2)}\ldots b_{\pi(m)}a_{\pi(m)}b_{\pi(1)}$ is clearly an alternating Hamilton cycle in K_n^c, as needed.

In case $n = 2m+1$ is odd we fix a path $P = a_1c_1b_1$ of length 2, so that the edges a_1c_1 and c_1b_1 have distinct colors, choose a random perfect matching $a_2b_2, \ldots, a_mb_m$ in the rest of the graph and show that with high probability there is an alternating Hamilton cycle containing the path P and the matching by applying Theorem 2.4 as before. Since the details are almost identical to the ones for the even case, we omit them. This completes the proof of the theorem. □

3 Concluding remarks, extensions and problems

1. Chen and Daykin [8] proved that if $K_{m,m}^c$ is an edge colored complete bipartite graph with partite sets of cardinality m each and $\Delta(K_{m,m}^c) \leq m/25$ then $K_{m,m}^c$ contains an alternating Hamilton cycle. Our proof of Theorem 1.1 contains a proof of the following;

   Proposition 3.1 *For every $\epsilon > 0$ there exists an $m_0 = m_0(\epsilon)$ so that for every $m > m_0$, every $K_{m,m}^c$ satisfying*

   $$\Delta(K_{m,m}^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)m \quad (\quad = (0.2928... - \epsilon)m \quad)$$

   contains an alternating Hamilton cycle.

2. The authors of [6] show that if $\Delta(K_n^c) < n/69$ then, in fact, K_n^c contains alternating cycles of all lengths from 3 to n. Similarly, in [8] the same conclusion is shown to follow from the weaker

assumption $\Delta(K_n^c) \leq n/17$. Our method here enables us to prove the following stronger result, which extends Theorem 1.1.

Theorem 3.2 *For every $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon)$ so that for every $n > n_0$, every K_n^c satisfying*

$$\Delta(K_n^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)n \quad (\quad = (0.2928... - \epsilon)n \quad)$$

contains alternating cycles of all lengths between 3 and n.

The proof is very similar to the proof of Theorem 1.1, but instead of Theorem 2.4 we need the following result, which is very similar to a result of Häggkvist and Thomassen (cf., e.g., [7]) and may be known. Since we were unable to find a reference we include a simple proof.

Lemma 3.3 *Any directed graph D on m vertices in which every indegree and every outdegree is at least $m/2 + 1$ is vertex pancyclic. That is, for every vertex v of D and every integer k between 2 and m, there is a directed cycle of length k through v.*

Proof. By Theorem 2.4 there is a Hamilton cycle $u_1 u_2 \ldots u_{m-1} u_1$ in $D - v$. Let N^+ and N^- be the sets of outneighbors and inneighbors of v, respectively. If there is no cycle of length k through v then for every i, $|N^+ \cap \{u_i\}| + |N^- \cap \{u_{i+k-2}\}| \leq 1$, where the indices are computed modulo $m - 1$. By summing over all values of i, $1 \leq i \leq m - 1$ we conclude that $|N^-| + |N^+| \leq m - 1$, contradicting the assumption that all indegrees and outdegrees exceed $m/2$. $\square$

Proof of Theorem 3.2. Consider, first, the case $n = 2m$. As in the proof of Theorem 1.1, given an edge colored K_n^c satisfying (1) apply Lemma 2.1 and Corollary 2.3 to obtain a matching $a_i b_i$ satisfying the two conditions in the corollary. Construct a digraph $D = (V, E)$ on the set of vertices $V = \{v_1, v_2, \ldots, v_m\}$ by letting $v_i v_j$ be a directed edge (for $i \neq j$) iff the color of $a_i b_j$ in K_n^c differs from that of $a_i b_i$ and that of $a_j b_j$. By Corollary 2.3 every indegree and every outdegree of D is at least $m/2 + 1$, implying, by Lemma 3.3, that D contains a directed cycle of every length between 2 and m. This gives, as in the proof of Theorem 1.1, that K_n^c contains an alternating cycle of each *even* length between 4 and n. To get the odd cycles we argue as follows. The expected number of pairs of edges with the same color in a randomly chosen triangle is less than 1, proving the existence of an alternating triangle. For the larger odd lengths we first fix a path $P = a_1 c_1 b_1$ of length 2, so that the edges $a_1 c_1$ and $c_1 b_1$ have distinct colors, choose a random perfect matching $a_2 b_2, a_3 b_3, \ldots, a_{m-1} b_{m-1}$ in the graph $K_n^c - \{a_1, b_1, c_1, v\}$, where $v \notin \{a_1, b_1, c_1\}$, and define a directed graph D on the vertices $v_1, v_2, v_3, \ldots, v_{m-1}$ in which the edges are defined as follows. For $i, j > 1$, $i \neq j$, $v_i v_j$ is a directed edge iff the color of $a_i b_j$ in K_n^c differs from that of $a_i b_i$ and that of $a_j b_j$. For $j > 1$, $v_1 v_j$ is a directed edge if the color of $a_1 b_j$ differs from that of $a_1 c_1$ and that of $a_j b_j$, whereas $v_j v_1$

is a directed edge if the color of $a_j b_1$ differs from that of $a_j b_j$ and that of $c_1 b_1$. As in the proof of Theorem 1.1 one can show that with positive probability every indegree and every outdegree in D exceeds $(m-1)/2+1$ and hence, by Lemma 3.3, for every k between 2 and $m-1$ D contains a cycle of length k through v_1. This cycle easily gives an alternating cycle of length $2k+1$ in K_n^c. Note that it is crucial to choose a cycle through the vertex v_1 here. The case $n = 2m+1$ is proved similarly. It is worth noting that the proof (with a slight modification) in fact shows that every edge of K_n^c is contained in an alternating cycle of each desired length between 4 and n (but not necessarily of length 3). $\square$

3. The last proof clearly contains a proof of the following extension of Proposition 3.1.

Proposition 3.4 *For every $\epsilon > 0$ there exists an $m_0 = m_0(\epsilon)$ so that for every $m > m_0$, every $K_{m,m}^c$ satisfying*

$$\Delta(K_{m,m}^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)m \quad (\quad = (0.2928... - \epsilon)m \quad)$$

contains an alternating cycle of every even length between 4 and $2m$.

4. Finally, it would be interesting to decide if the conjecture of [6] that asserts that if $\Delta(K_n^c) < \lfloor n/2 \rfloor$ then K_n^c contains an alternating Hamilton cycle is correct.

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References

[1] N. Alon, The strong chromatic number of a graph, *Random Structures and Algorithms* **3** (1992), 1-7.

[2] N. Alon and J. H. Spencer, *The Probabilistic Method*, Wiley, New York, 1992.

[3] J. Bang-Jensen and G. Gutin, Alternating cycles and paths in edge-coloured multigraphs: a survey, *Discrete Math.* **165-166** (1997), 39-60.

[4] J. Bang-Jensen, G. Gutin and H. Li, Sufficient conditions for a digraph to be Hamiltonian, *J. Graph Theory* **22** (1996), 181-187.

[5] C. Berge, *Graphs and Hypergraphs* (second edition), North Holland, Paris, 1976.

[6] B. Bollobás and P. Erdős, Alternating Hamiltonian cycles, *Israel J. Math.* **23** (1976), 126-131.

[7] J.-C. Bermond and C. Thomassen, Cycles in digraphs: a survey, *J. Graph Theory* **5** (1981), 1-43.

[8] C. C. Chen and D. E. Daykin, Graphs with Hamiltonian cycles having adjacent lines different colors, *J. Combin. Th. Ser. B* **21** (1976), 135-139.

[9] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Mathematics 1200, Springer Verlag, Berlin and New York, 1986.

[10] J. Shearer, A property of the colored complete graph. *Discrete Math.* **25** (1979), 175-178.