Classification of Bipartite Boolean Constraint Satisfaction through Delta Matroid Intersection

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Schaefer’s Classification Problems

Domain: \{0,1\}
Constants: 0,1

Polynomial Cases

- Horn clauses: \( \bar{x} \lor y \lor z, \ x \lor \bar{y} \lor \bar{z} \)
  closure function \( f(0,0) = f(0,1) = f(1,0) = 0 \quad f(1,1) = 1 \)

- Anti-Horn clauses: \( x \lor y \lor \bar{z}, \ x \lor y \lor z \)
  closure function \( f(0,0) = 0 \quad f(0,1) = f(1,0) = f(1,1) = 1 \)

- 2-satisfiability: \( x \lor y, \ x \lor \bar{y}, \ \bar{x} \lor \bar{y} \)
  closure function \( g(x, y, z) = \text{majority} (x, y, z) \)

- Linear equations modulo 2: \( x + y + z = 0 \pmod{2}, \ x + y + z = 1 \pmod{2} \)
  closure function \( h(x, y, z) = x + y + z \pmod{2} \)

All other problems are NP-complete

- 3-satisfiability: \( \bar{x} \lor y \lor z, \ \bar{x} \lor \bar{y} \lor \bar{z}, \ x \lor \bar{y} \lor \bar{z} \)

- One-in-3-satisfiability: \( \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\} \)

- Not-all-equal satisfiability: \( \{(x, y, z) \mid (0, 0, 0), (1, 1, 1)\} \)
What happens to NP-complete problems when restricted to two occurrences per variable?

- One-in-3-satisfiability
  
  \[
  \begin{array}{ccc}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  \end{array}
  \]
  graph matching

- Not-all-equal satisfiability
  
  \[
  \begin{array}{c}
  1, 2 \\
  \end{array}
  \]
  graph matching

- 3-satisfiability
  
  polynomial delta-matroid parity

All three become polynomial delta-matroid parity problems !!!
Towards a classification with two occurrences per variable

1. If not in Schaefer’s polynomial cases then can simulate all clauses
   
   $x \lor y \lor z, \quad x \lor y \lor z \lor t$

2. If not delta-matroid then can simulate

   $$R = (x \equiv y, z): \quad (0, 0, 0), (1, 1, 1) \in R \quad (x, y, z) \in R \Rightarrow x \leq y, z$$

1. and 2. simulate satisfiability with three occurrences per variable

   NP-complete !!!
Bipartite case classification with two occurrences per variable

One-in-three satisfiability $\rightarrow$ graph matching $\rightarrow$ bipartite graph matching

One left constraint and one right constraint
Delta-matroid parity $\rightarrow$ delta-matroid intersection

**Delta-matroids:**

$[n] = \{1, 2, \ldots, n\}$
$B = \{A \subseteq [n]\} \text{ bases}$

$A_1 \in B, A_2 \in B, i \in (A_1 \Delta A_2) \Rightarrow \exists j \in (A_1 \Delta A_2) : (A_1 \Delta \{i, j\}) \in B$

**Bases:**

$i \in A: \quad x_i = 1$

$i \notin A: \quad x_i = 0$
Delta-matroid intersection more general than delta-matroid parity

if = delta-matroid \([x = y] = \{(0, 0), (1, 1)\}\) not allowed or
not = delta-matroid \([x \not= y] = \{(0, 1), (1, 0)\}\) not allowed
if = disallowed: simple blossom augmentation gives polynomial algorithm
if not = disallowed: simple blossom augmentation gives polynomial algorithm
if =, not = disallowed in one delta-matroid, other arbitrary: simple blossom augmentation gives polynomial algorithm

simple blossom collapses and does augmentation in these cases
Bipartite classification with oracles

1. NP-complete cases

2. Schaefer-derived cases:
   - Horn clauses, anti-Horn clauses, 2-satisfiability, linear equations modulo 2
   - One side has only monadic constraints
   - Upward closed 2-sat in one side, other side 2-sat downward closure (or vice versa)

3. 2-sat upward closed and delta-matroid downward closed in one side, reverse in the other side

4. Delta-matroid derived cases:
   - Delta-matroid intersection without equality
   - Delta-matroid intersection without equality, inequality in one side
   - Upward delta-matroid in one side, downward closure of other side is delta-matroid
   - Delta matroid parity with equality

Open!!!
   - Local odd and even delta-matroids
   - Local-zebra and linear-zebra delta-matroids (not with oracle)
   - Delta-matroid without inequality
   - etc.
Open problems

* Zebra cases do not work with oracles, but polynomial. Other such problems?
  Linear delta-matroids seem to work with oracles and not just linear representation.

* $k$-partite for $k \geq 3$: solved classification, polynomial with oracles or NP-complete

* Multi-domain case: $\{0_A, 1_A\}, \{0_B, 1_B\}…$
  With 2 (or more) occurrences per variable, solved when relations satisfy symmetry:
  exchanging first and second occurrences of variables gives another valid relation.
  What about without symmetry?
  List constraints have been classified when subsets of lists of size at most 3 are also lists.
  What about if only subsets of size at most 2 are required lists. Also are NP-complete cases
  still provable with 3 occurrences, and what about 2 occurrences?