An(other) algebraic theory of the context-free languages?

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July 2013
FSMNLP St Andrews

Outline

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FSMNLP

1. Regular languages are awesome.
2. Natural languages aren't regular.

Regular languages

Classical Automata Theory

- Equivalence of regular expressions, DFAs and NFAs
- Myhill-Nerode theorem
- Canonical representations: minimal DFA

Algorithms

- Determinisation and Minimisation algorithms
- FST and weighted FST libraries
- Angluin's learnability results

Overview

Current NLP practice uses two ingredients:

1. A 50-year old representation:

   Finite Automata McCulloch and Pitts (1943)
   Categorial Grammars Bar-Hillel (1953)
   Context Free Grammars Chomsky (1956)
   Type-logical Grammars Lambek (1958)
   Dependency grammars Tesnière (1959)

2. Modern machine learning techniques:
   - Max-margin approaches and SVNs
   - Non-parametric Bayesian plus Markov Chain Monte Carlo
   - Deep belief networks
   - Spectral learning algorithms
   - ...

Goals

Questions

- Can we redo the theory of context-free languages to make it as useful as the theory of regular languages?
- And can we do the same later for the mildly context-sensitive language hierarchy?
- Can we use this to design algorithms for learning, minimizing, determinising, testing equivalence, . . . of context free grammars?

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- Can we redo the theory of context-free languages to make it as useful as the theory of regular languages?
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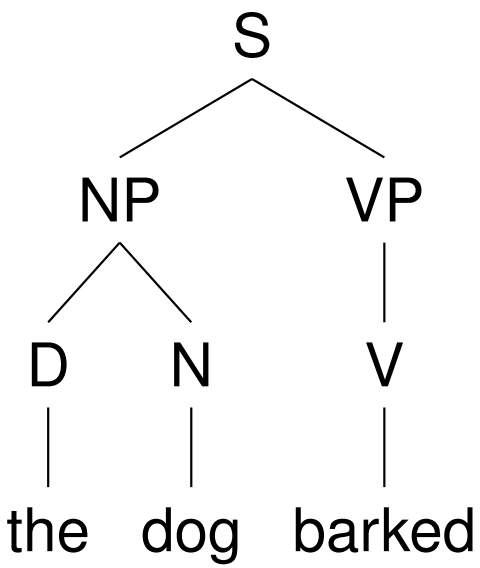
Maybe.

A naive question in linguistics



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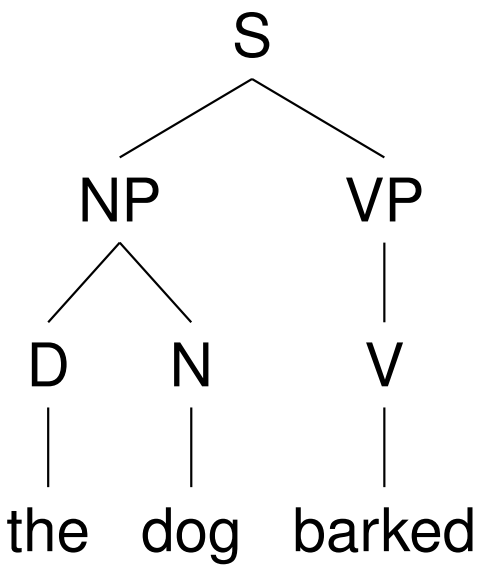

Question

What does this mean? What do the symbols mean?

Definitions

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A tuple $\langle \Sigma, V, S, P \rangle$, where

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- $P \subset V \times (V \cup \Sigma)^+$ is a set of productions written $N \to \alpha$

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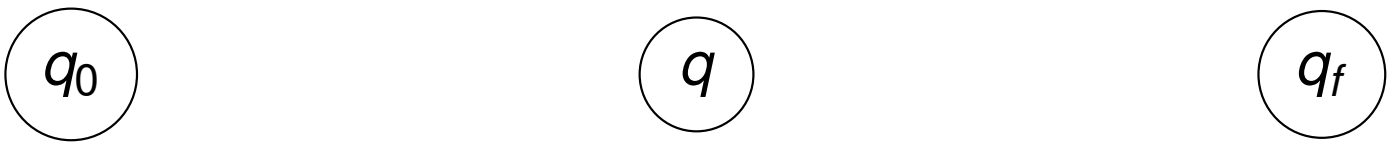
Questions

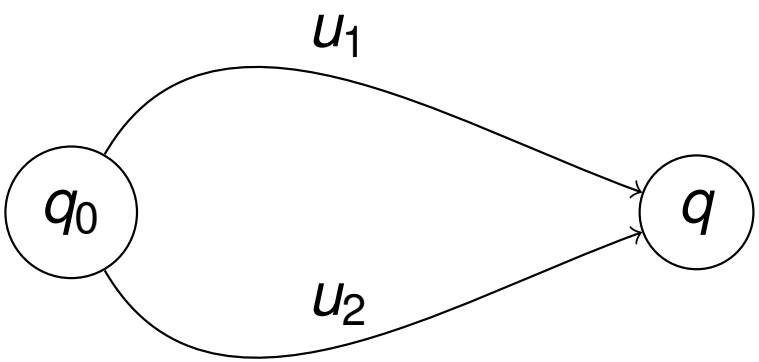
What are the elements of Σ? What are the elements of $V = \{N, P, Q \dots\}$? What do they mean? What does S mean? What does a production like $N \to aPQ$ mean?

Outline

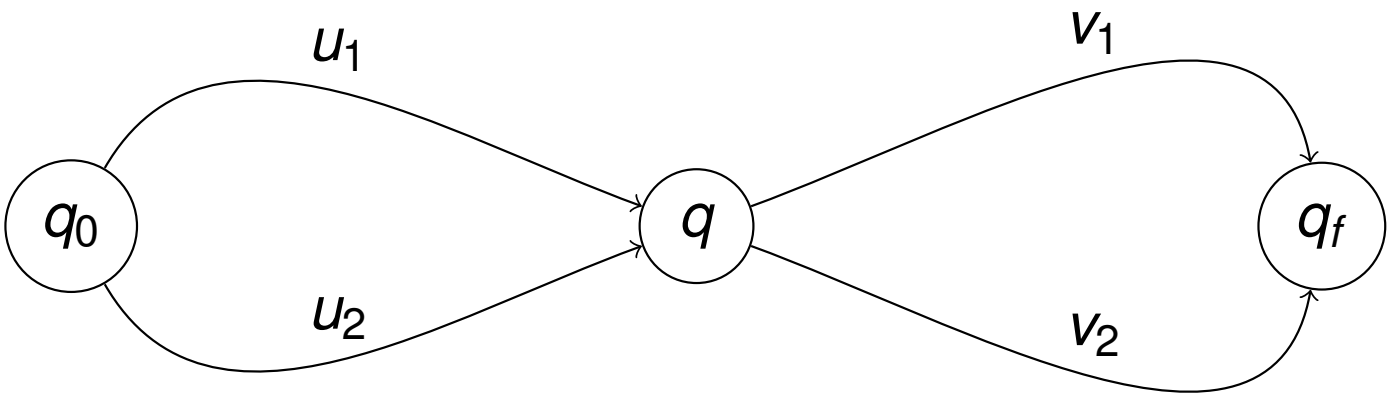
Finite automata



Finite automata



Finite automata



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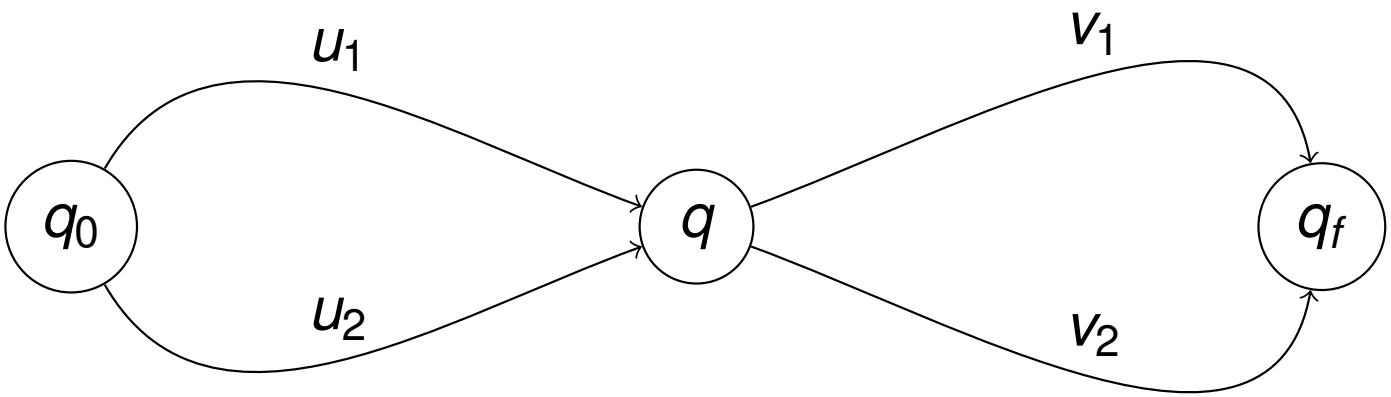


$$\{u_1, u_2, \ldots\} \circ \{v_1, v_2, \ldots\} \subseteq L$$

Decompositions

For a state q

$P_q \circ S_q \subseteq L$

Decompositions

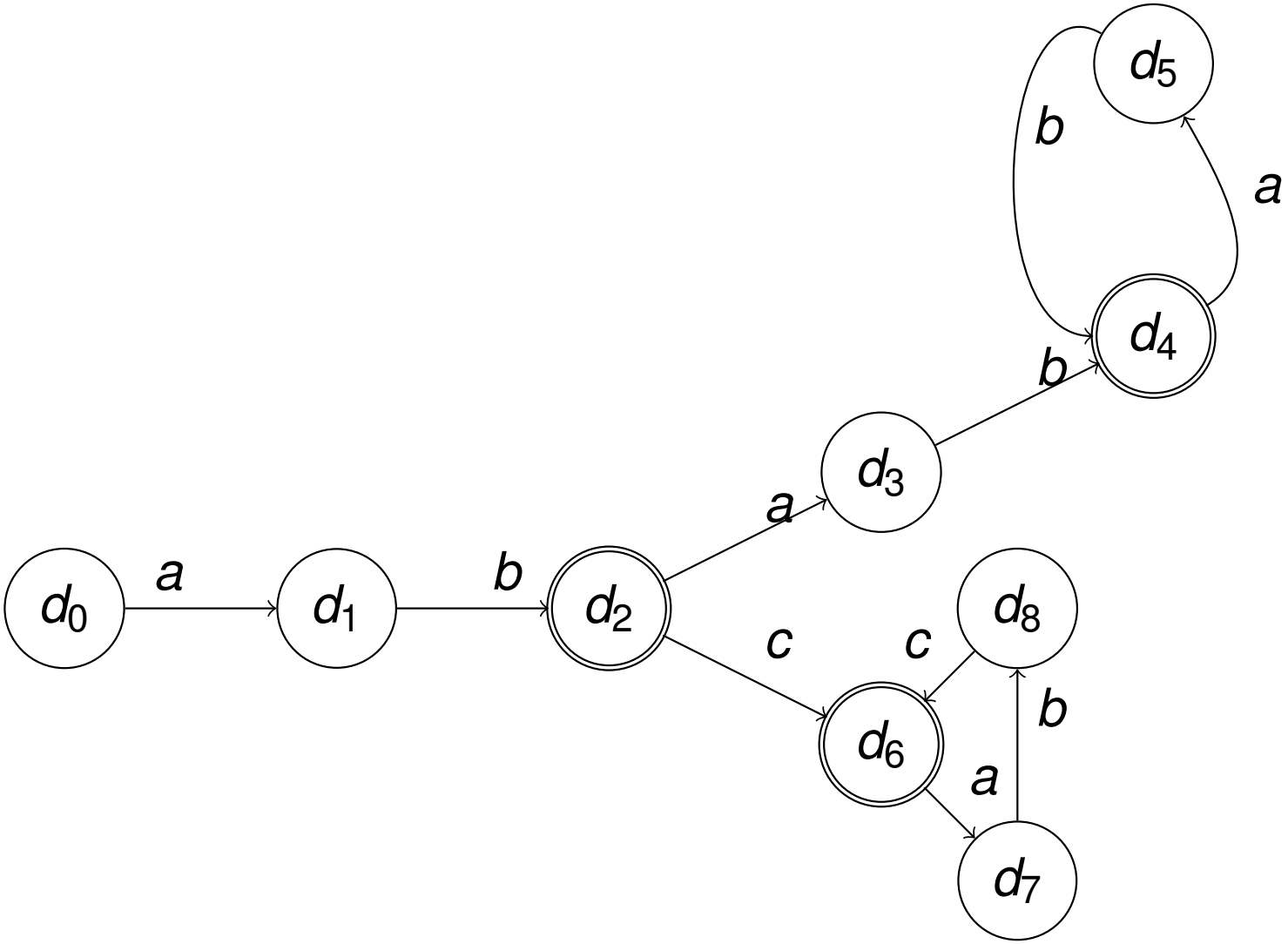
For a state q

$P_q \circ S_q \subseteq L$

- If $\lambda \in P_q$ then it is a start state
- If $\lambda \in S_q$ then it is an end state
- If $u \in P_q$ and $ua \in P_{q'}$ then there is a transition $q \rightarrow^a q'$

DFA

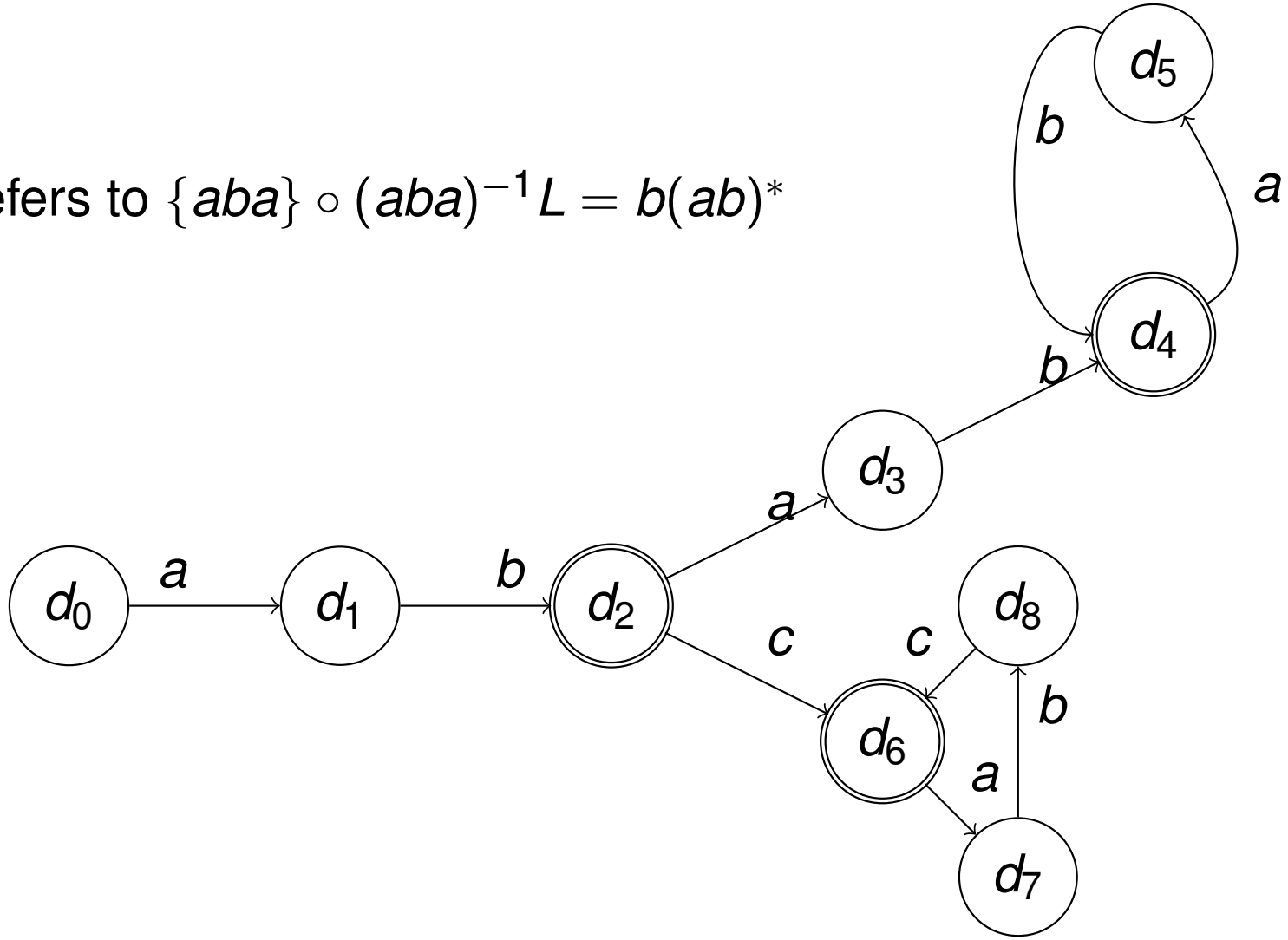
$(ab)^+ \cup (abc)^+$



DFA

$(ab)^+ \cup (abc)^+$

d_3 refers to $\{aba\} \circ (aba)^{-1}L = b(ab)^*$



Myhill-Nerode theorem

$$u^{-1}L = \{v \mid uv \in L\}$$

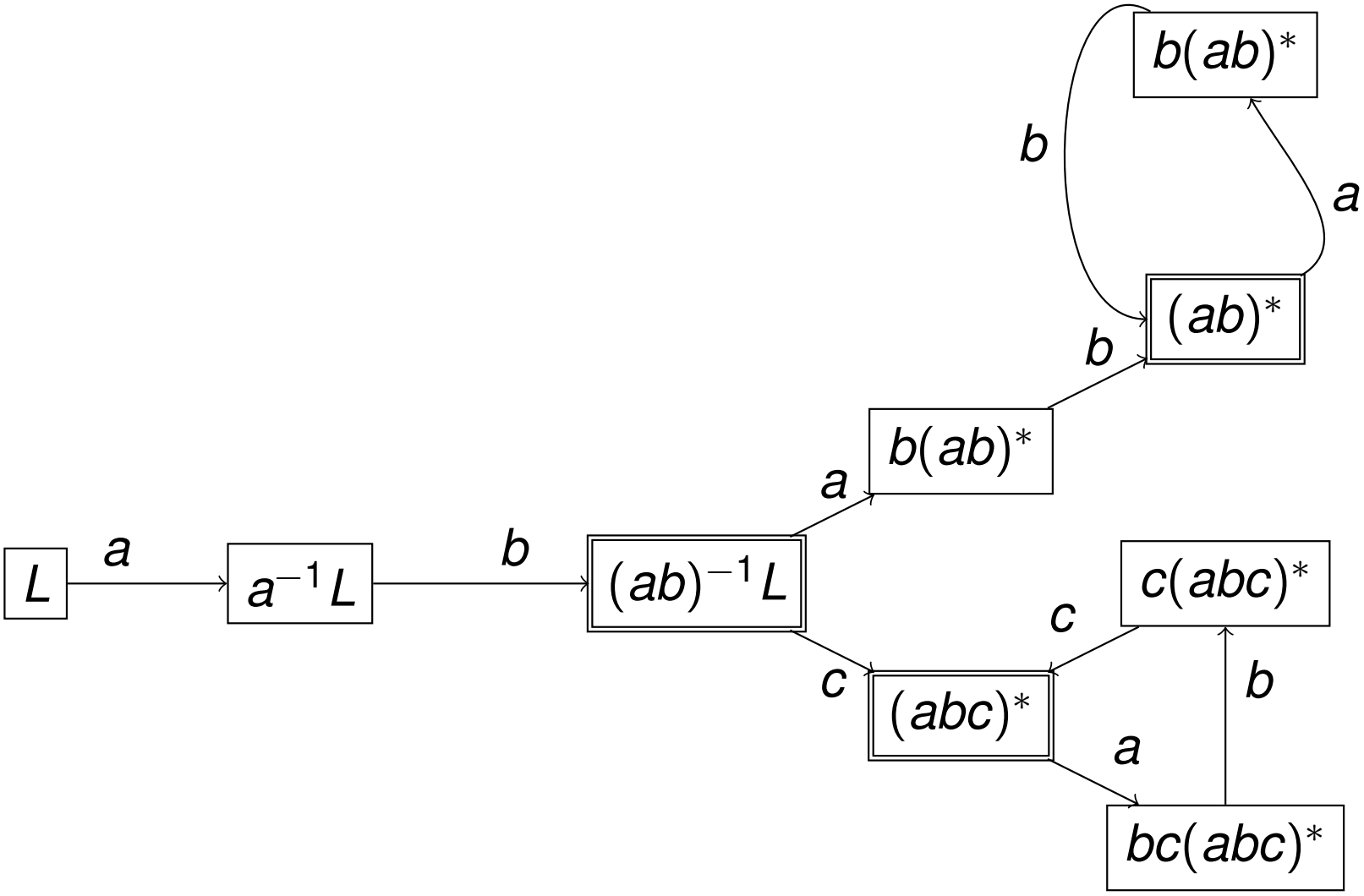
$$w_1 \sim_L w_2 \text{ iff } w_1^{-1}L = w_2^{-1}L$$

$$[u]_{\sim_L} = \{v \mid u^{-1}L = v^{-1}L\}$$

The number of equivalence classes of $\sim_L$ is equal to the number of states in the minimal DFA that accepts L.

Minimal DFA

$(ab)^+ \cup (abc)^+$



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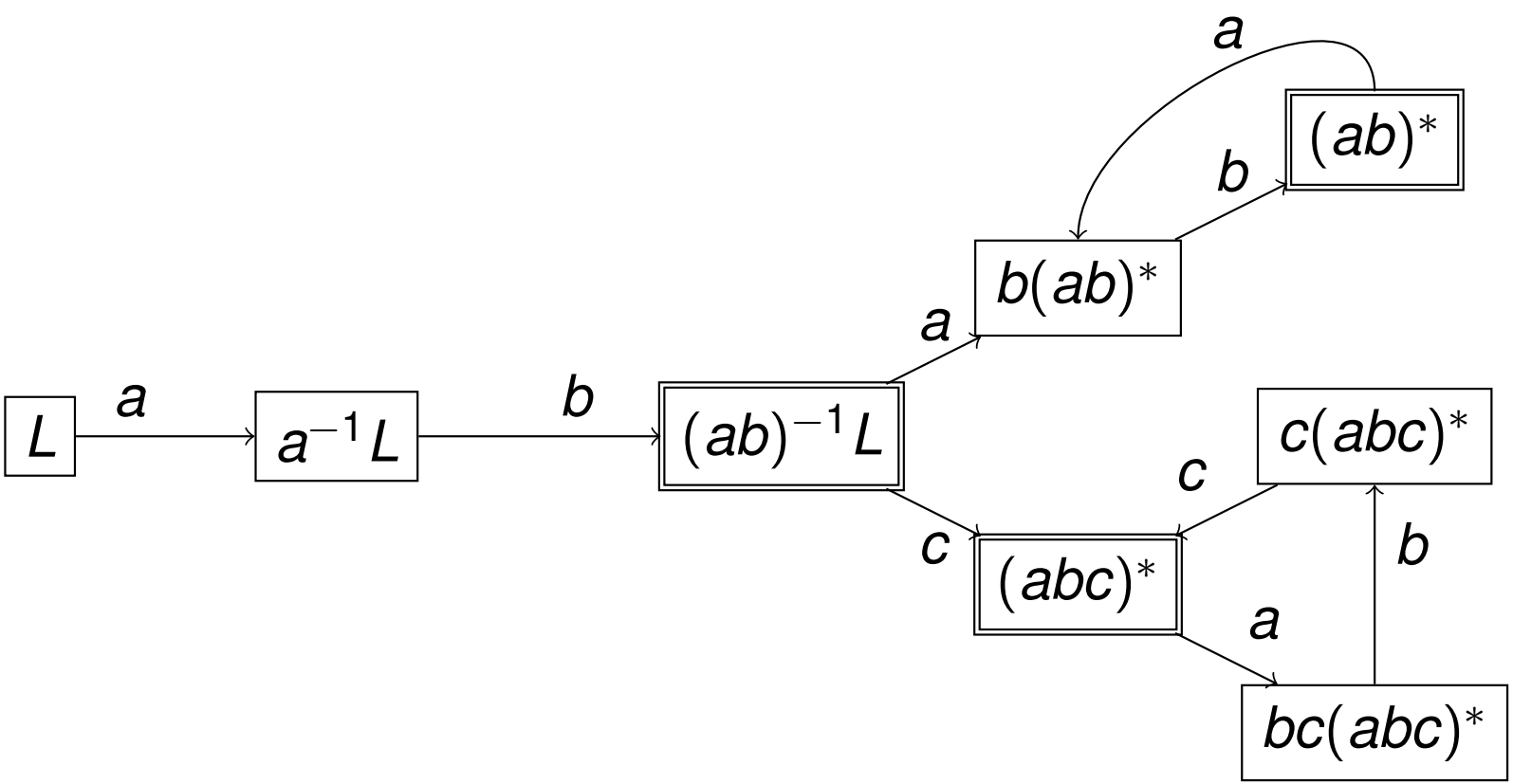

Nice properties of the Canonical DFA

Minimality

It is the smallest DFA that generates L.

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It is the smallest DFA that generates L.

Universality or “Maximality”

It is (contains) an exact morphic image of every DFA that generates L.

Morphisms of automata

Definition

A morphism from an automaton A to an automaton B is a function:
ϕ from Q_A to Q_B such that: for all $q \in Q_A$:

$$\phi(q_0^A) = q_0^B$$

$$\phi(F_A) \subseteq F_B$$

where defined $\delta_B(\phi(q), a) = \phi(\delta_A(q, a))$

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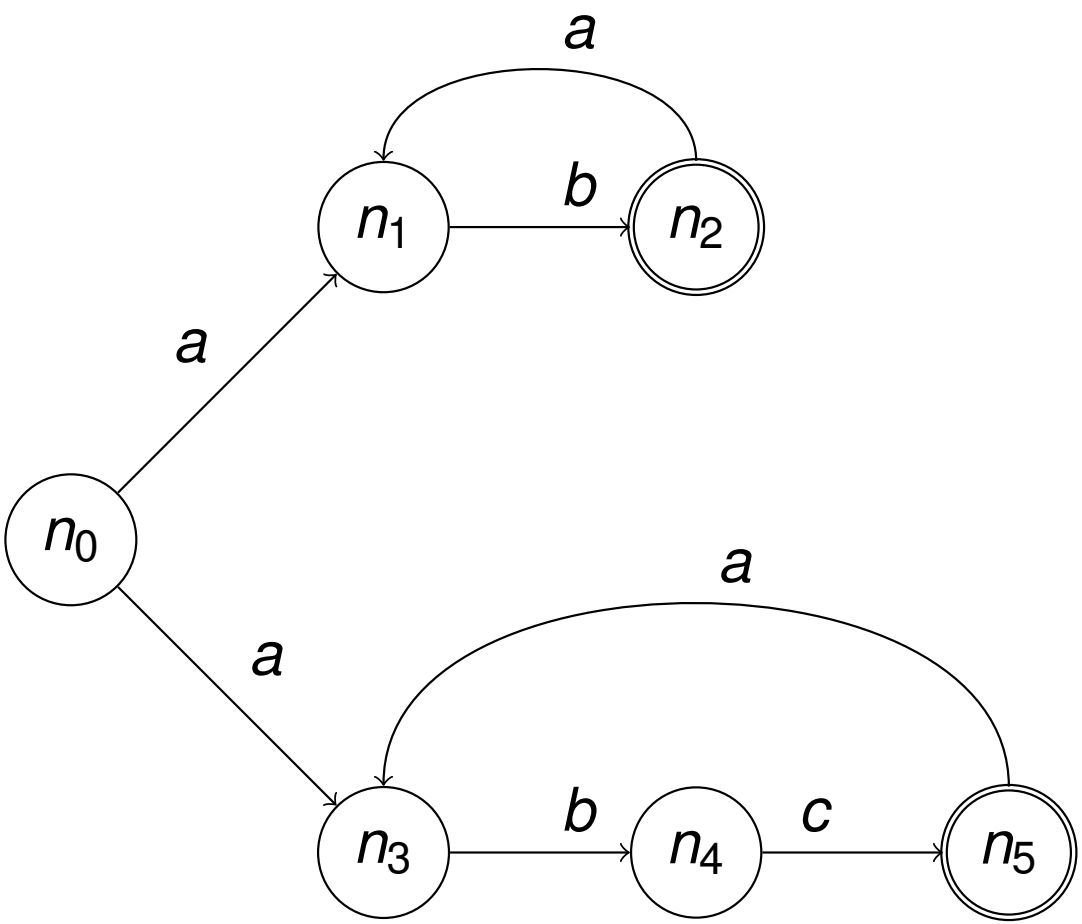
The morphism merges states p, q when $\phi(p) = \phi(q)$.

Morphism into the canonical DFA

- The canonical DFA has states which are decompositions of the form $[u]_{\sim_L} \circ u^{-1}L$.
- So it only has one state for each residual language and because the equivalence classes $[u]$ are a partition this is deterministic.
- Other DFAs will have states $P \circ u^{-1}L$ where $P \subset [u]$,
- Canonical morphism ϕ_L maps each state in a DFA to the unique state in the canonical DFA.

NFA

$(ab)^+ \cup (abc)^+$



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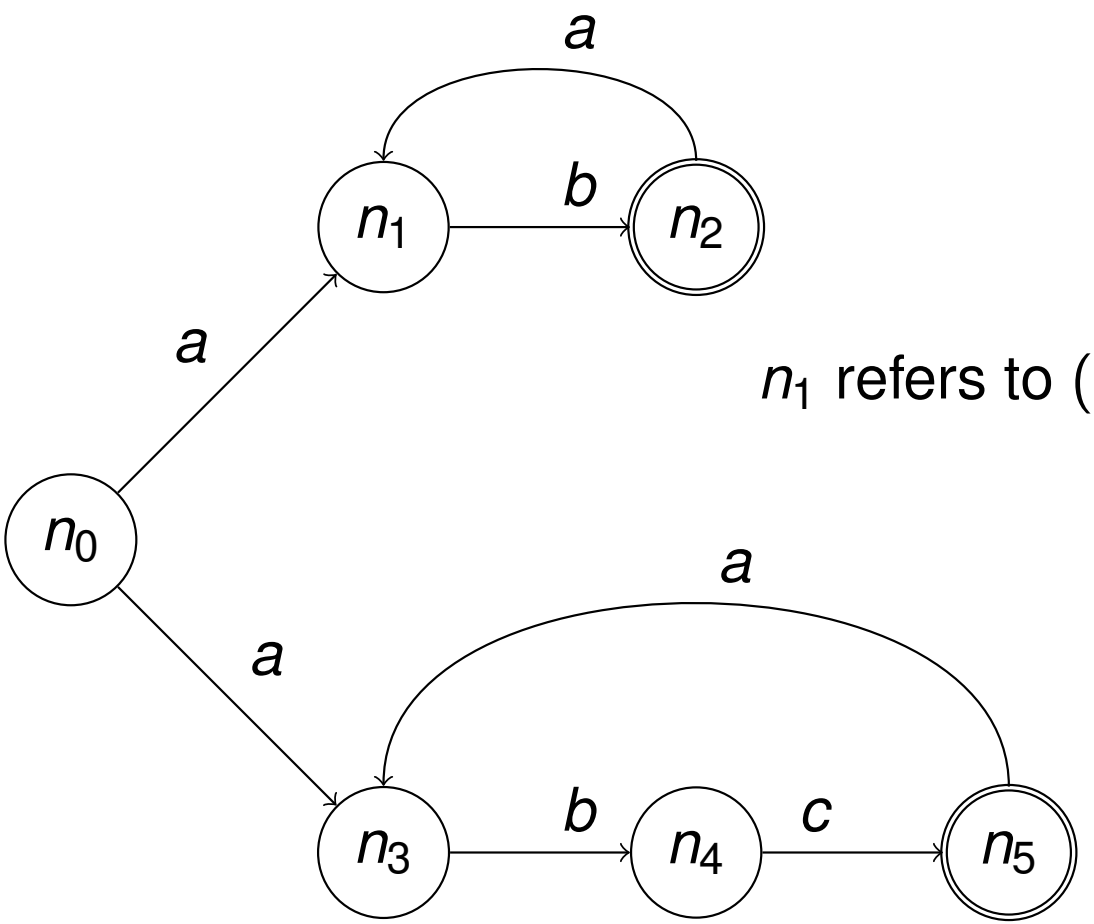
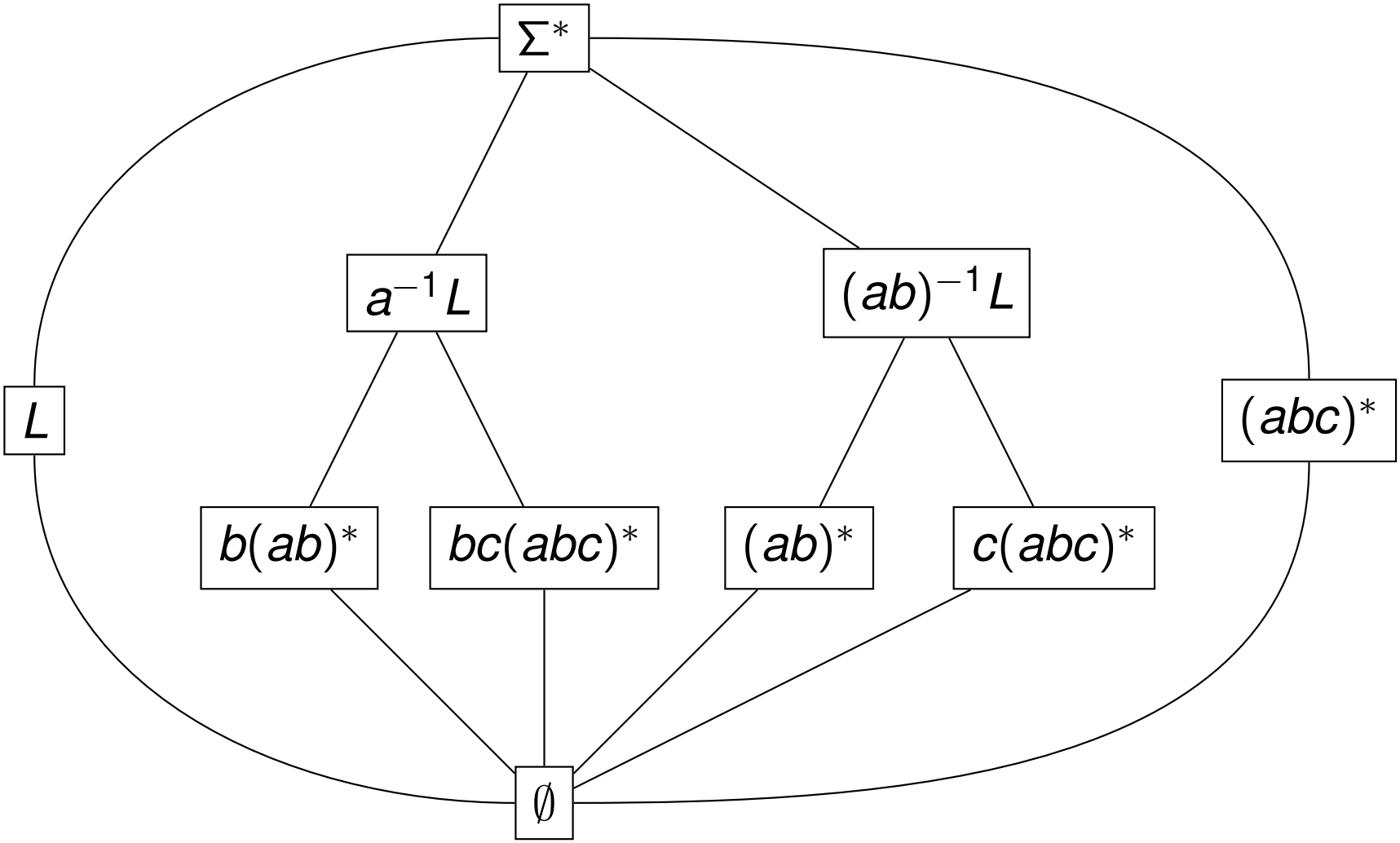


n_1 refers to $((ab)^* a) \circ (b(ab)^*)$

A hierarchy of maximum factorisations

Showing suffixes



Universal automaton

States

Maximal factorisations P, S such that $PS \subseteq L$.

Start states

States such that $\lambda \in P$.

Accepting states

States such that $\lambda \in S$

Transitions

Transition from $\langle P, S \rangle \rightarrow^a \langle P', S' \rangle$ iff $Pa' \subseteq P'$

Nice properties of the UA for a regular language L

Universality

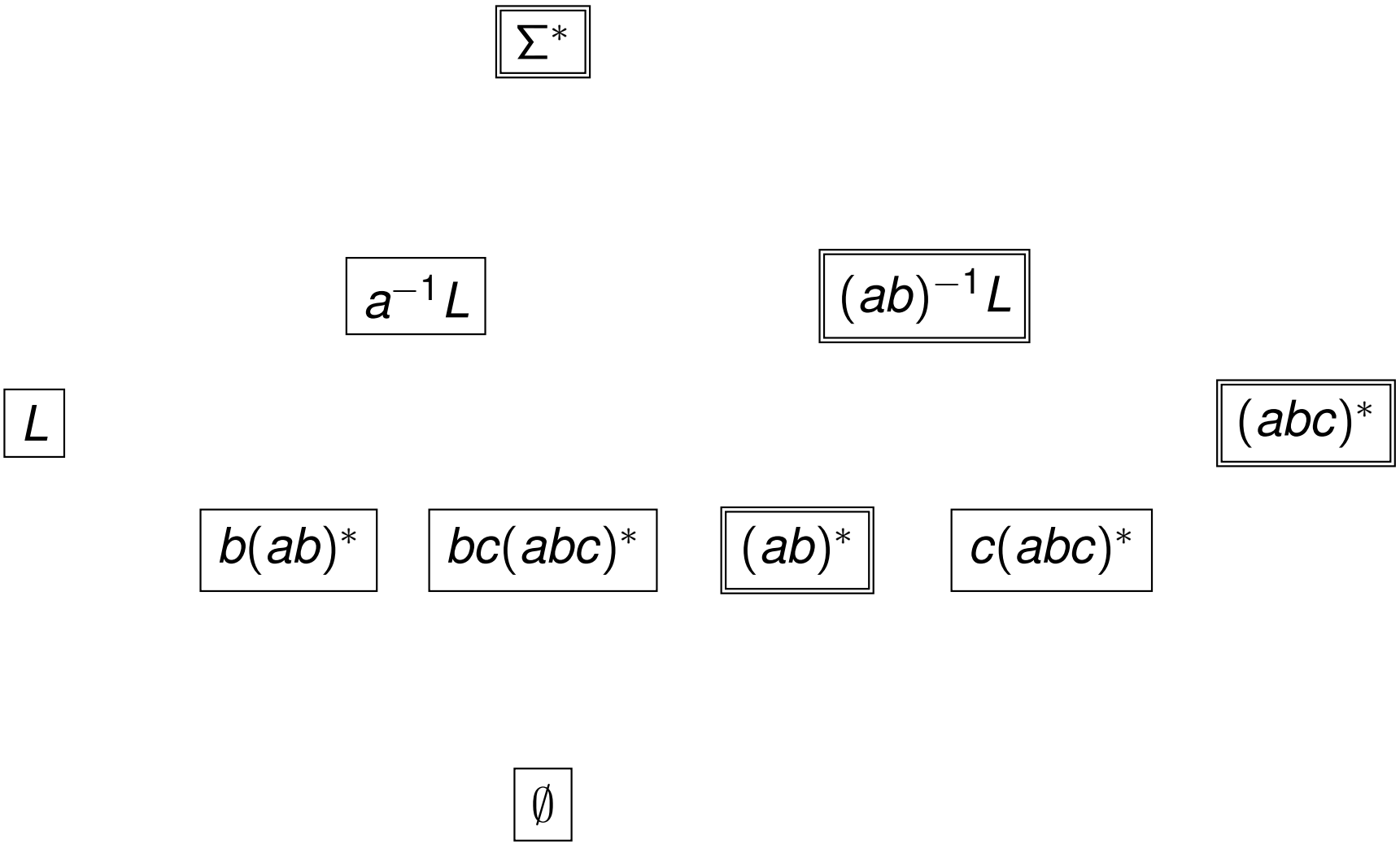
It contains a morphic image of every FA that generates L.

Minimality

It contains every minimal automaton as a subautomaton.

Hierarchy of states

$(ab)^+ \cup (abc)^+$



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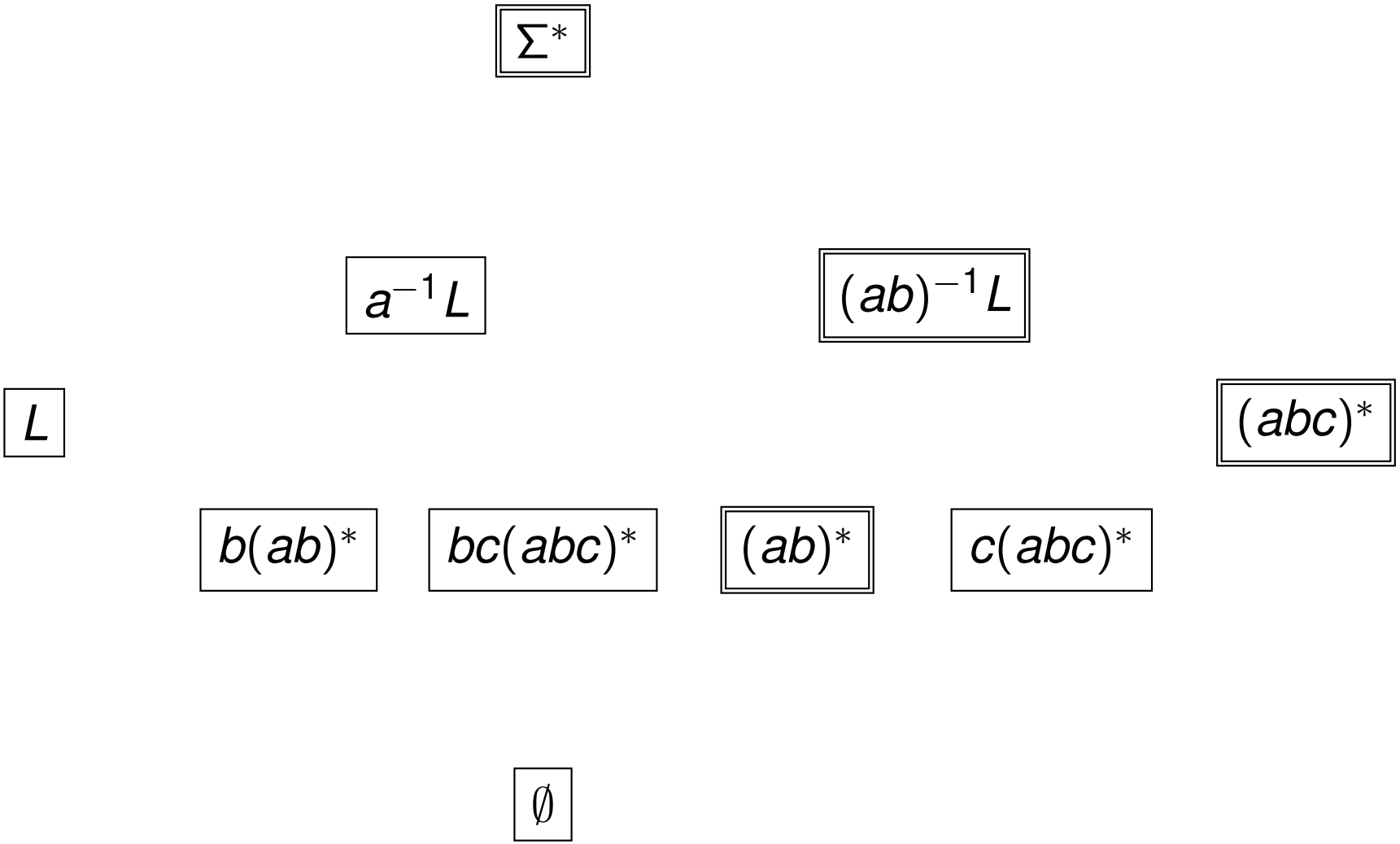

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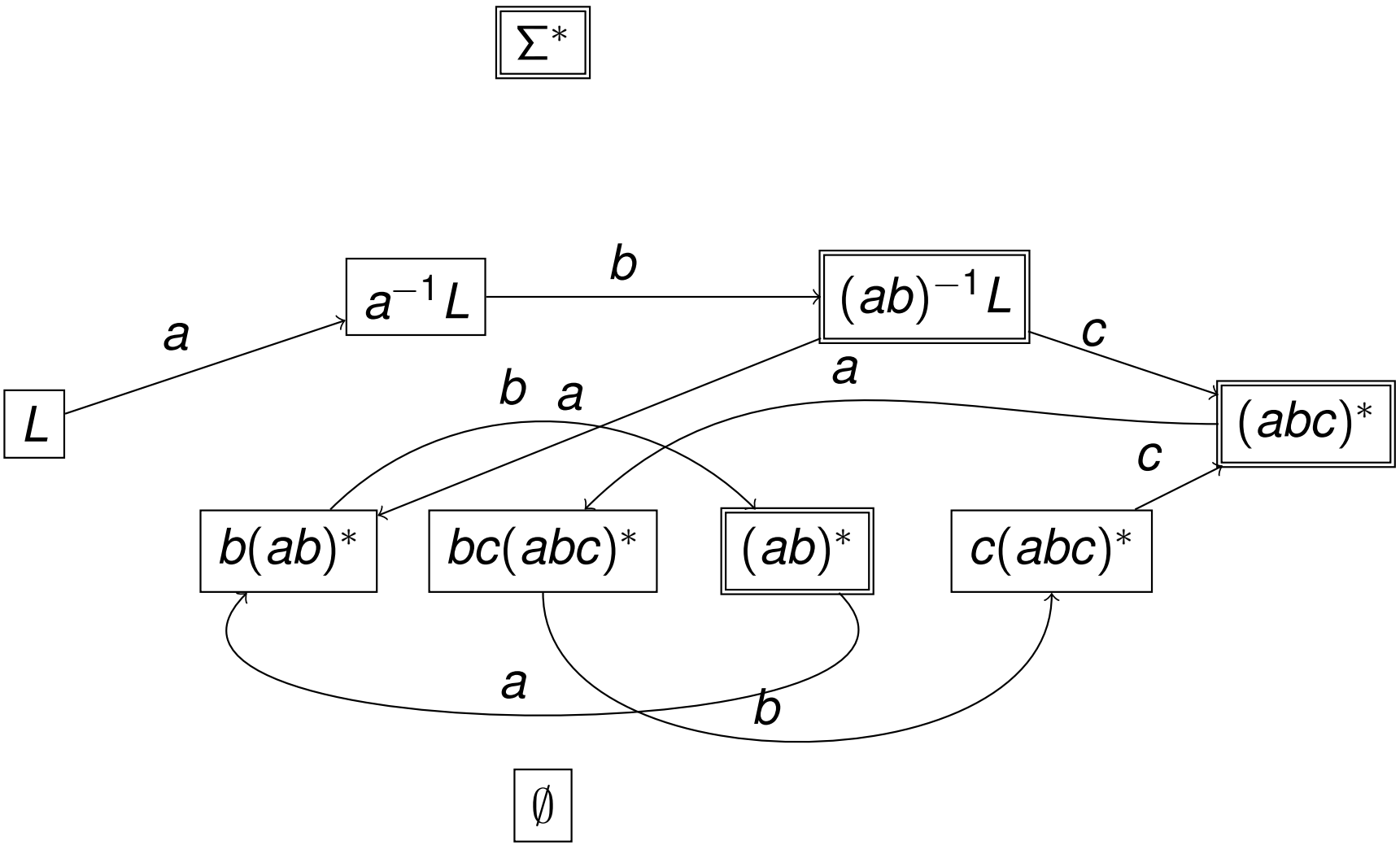

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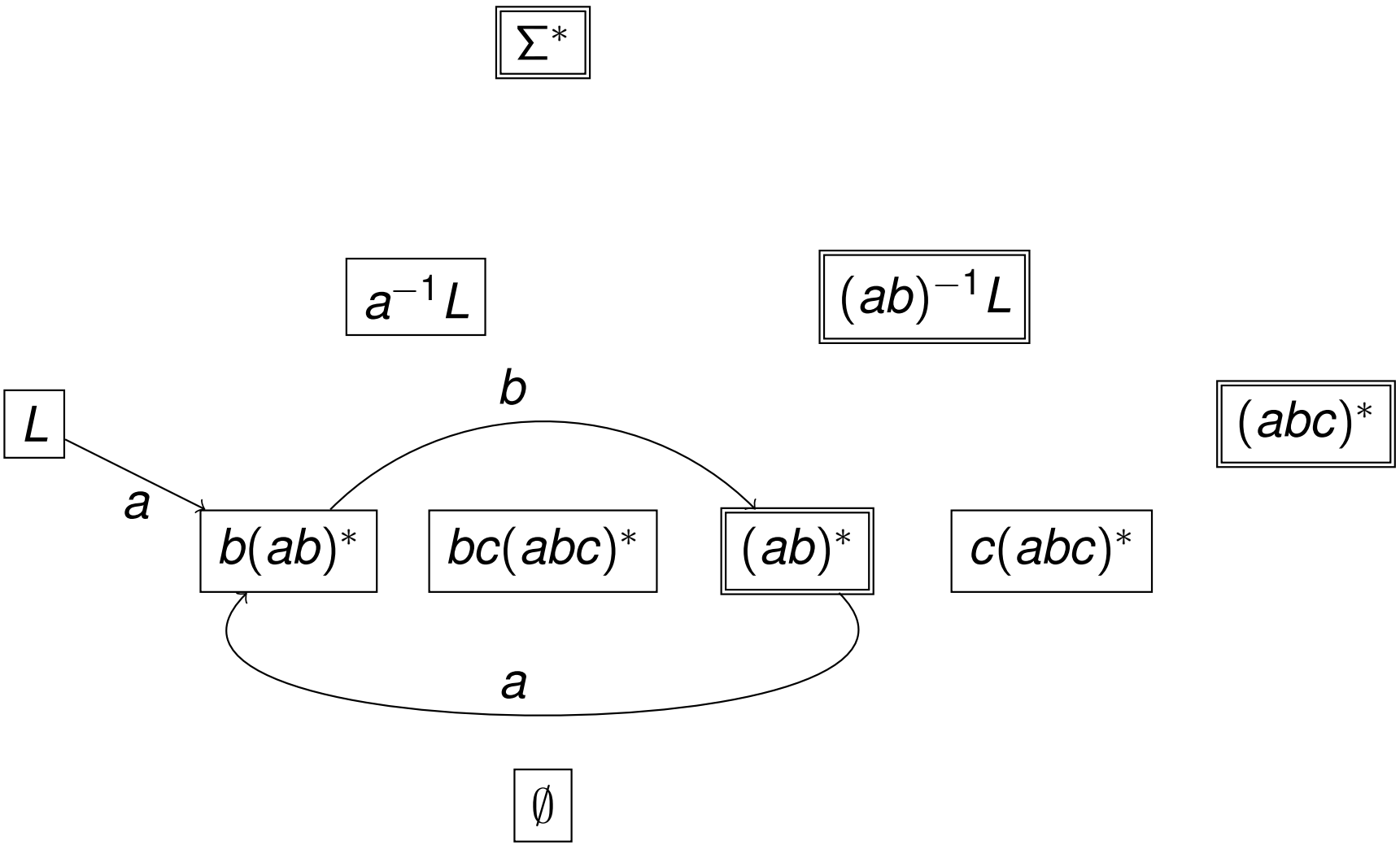

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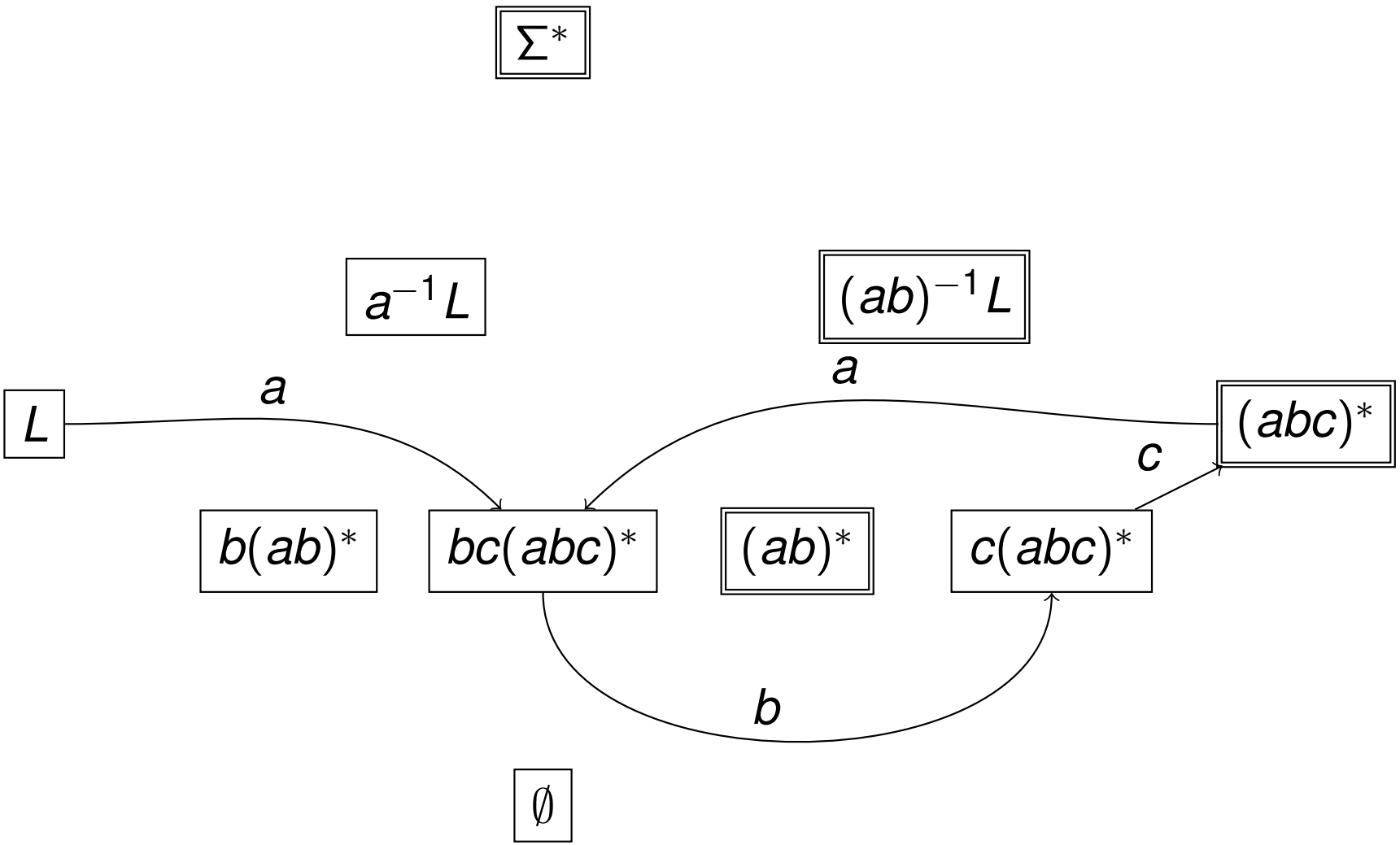

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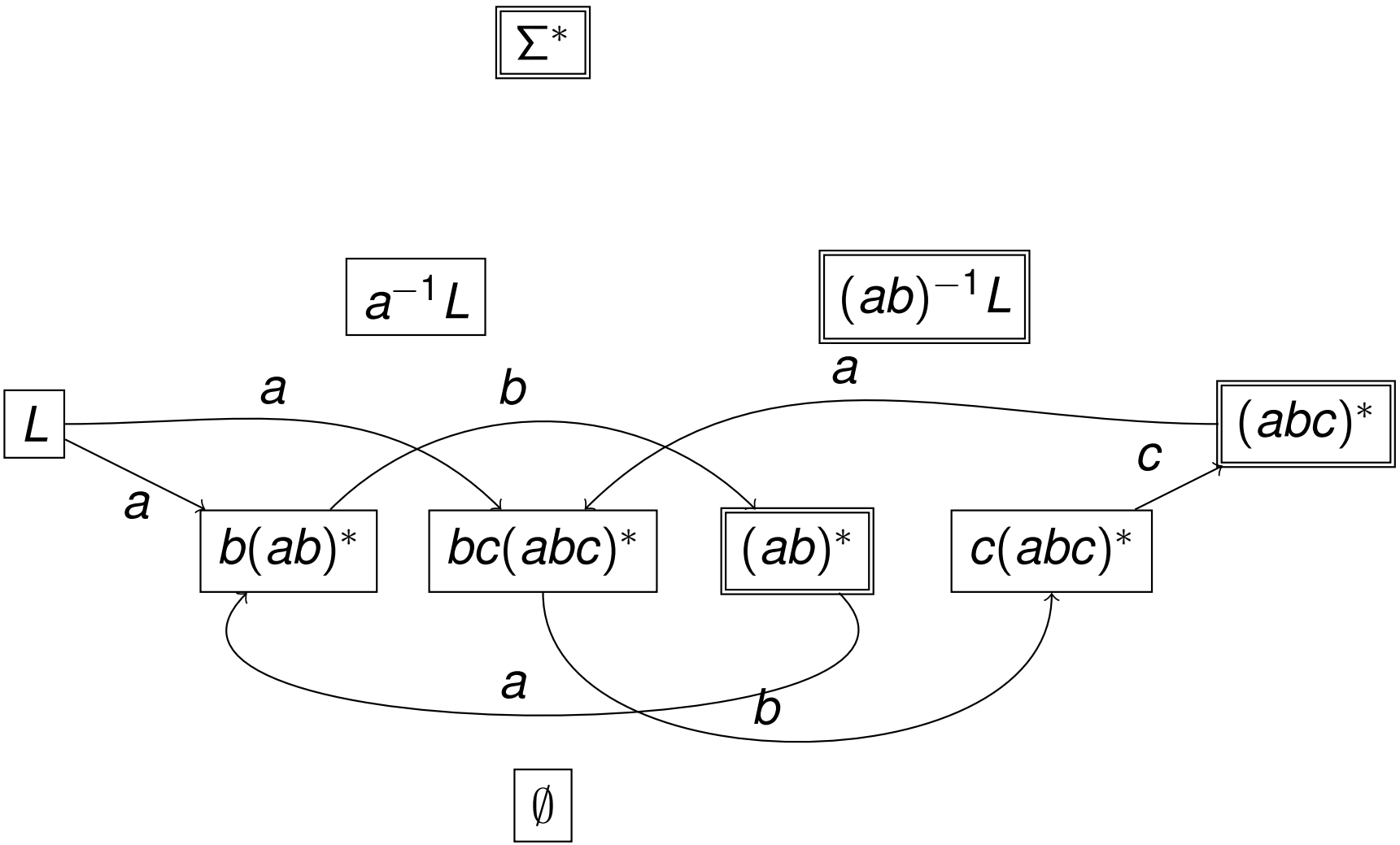

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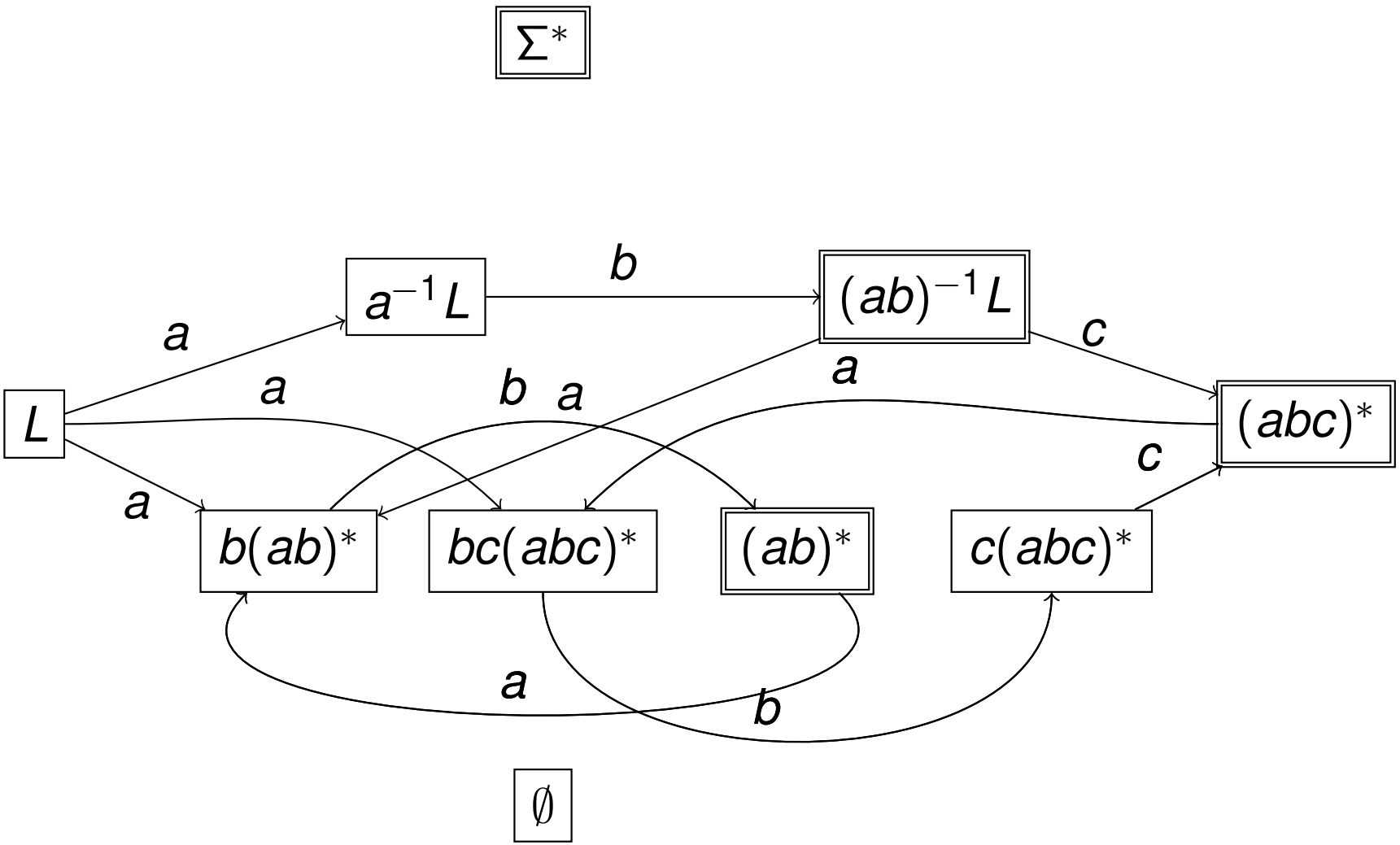

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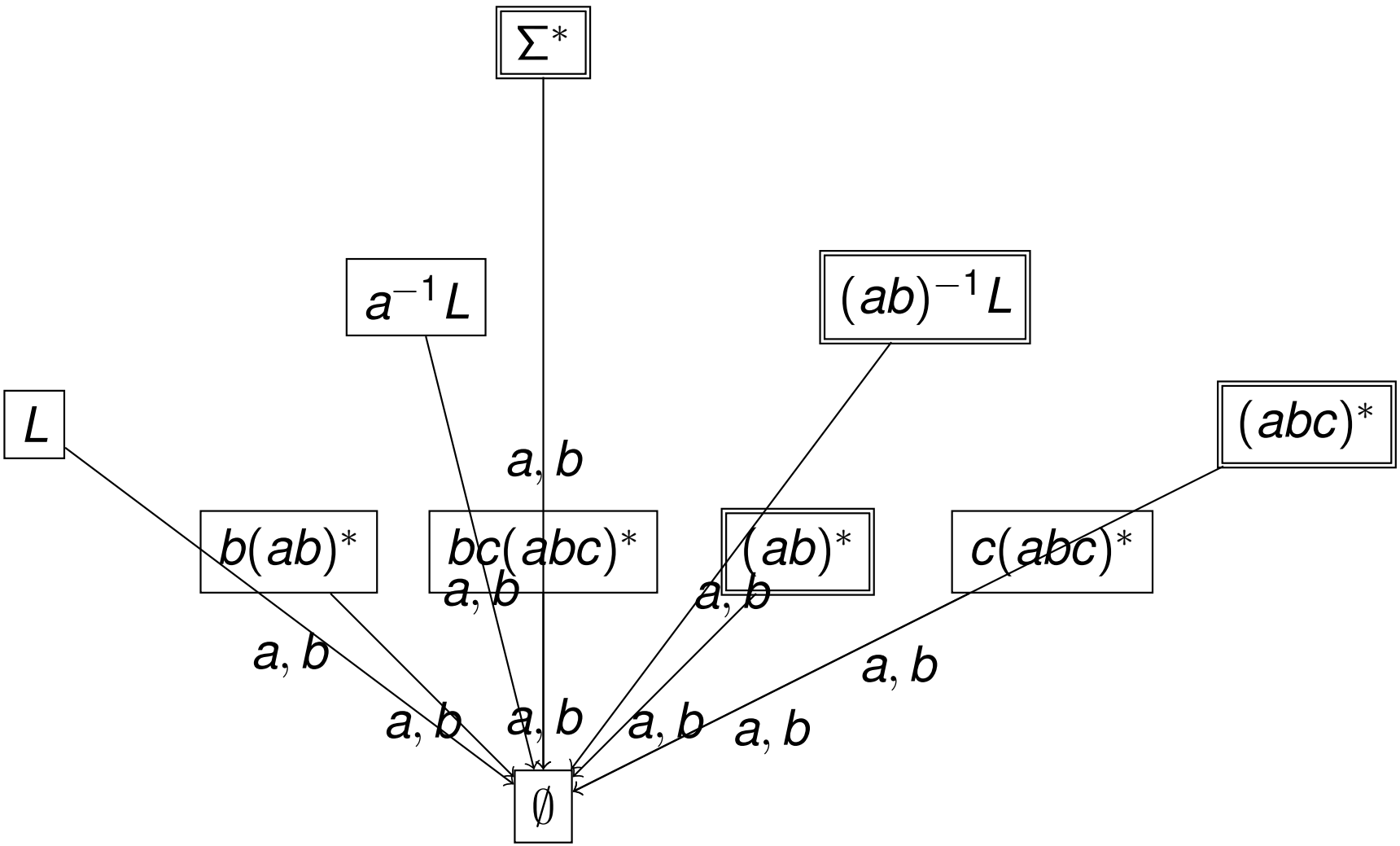

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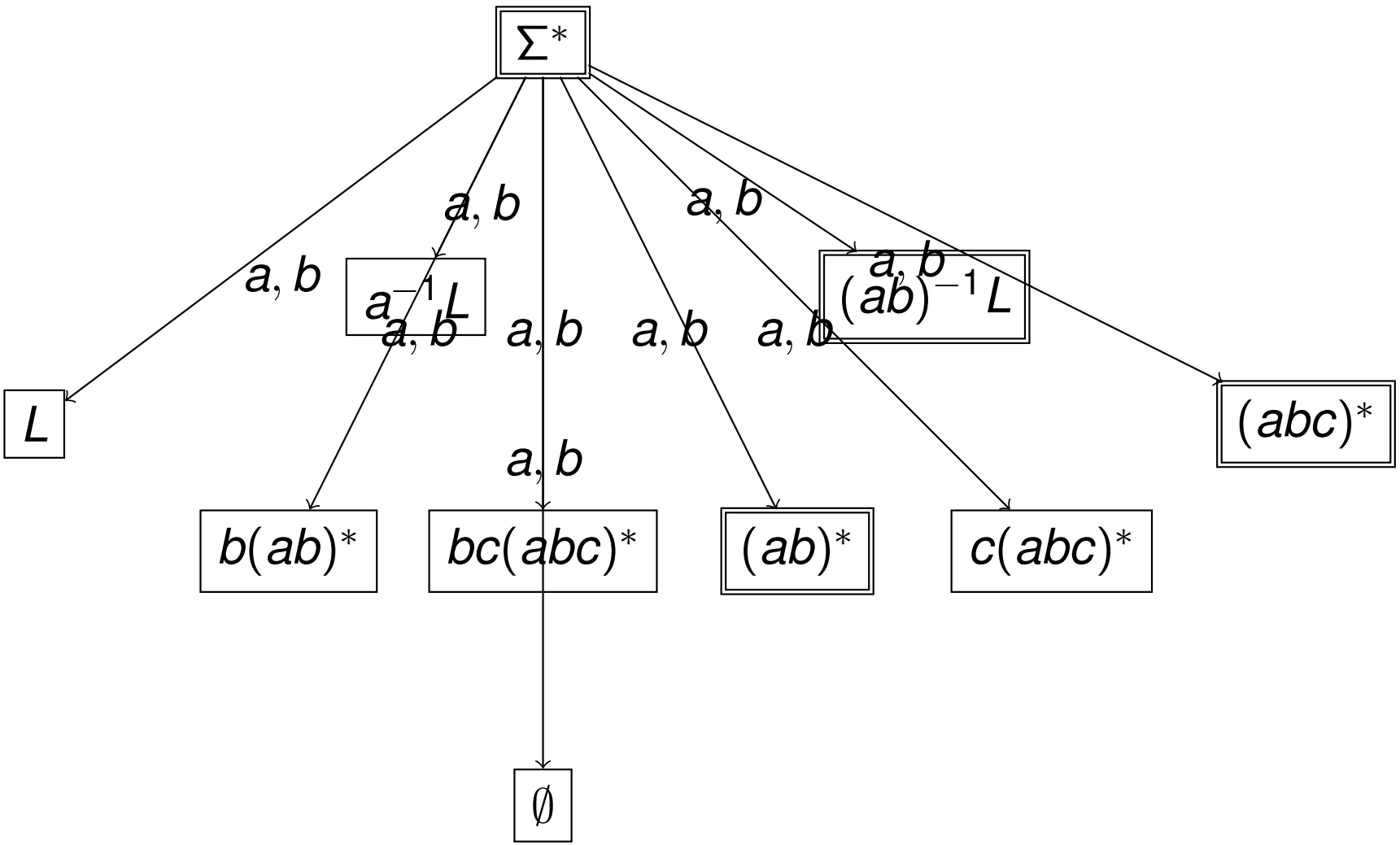

Regular languages

Question

What does the state in a DFA mean?
It is a residual language.

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What is the state in a regular grammar/NFA?
It is a maximal decomposition $P \circ Q \subseteq L$.

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States are decompositions

An automaton defines a decomposition of a particular type.
We can start with a *maximal* decomposition, and end with a *minimal* automaton.

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Semantics of Context-free languages I

Rewriting systems (Chomsky, 1959)

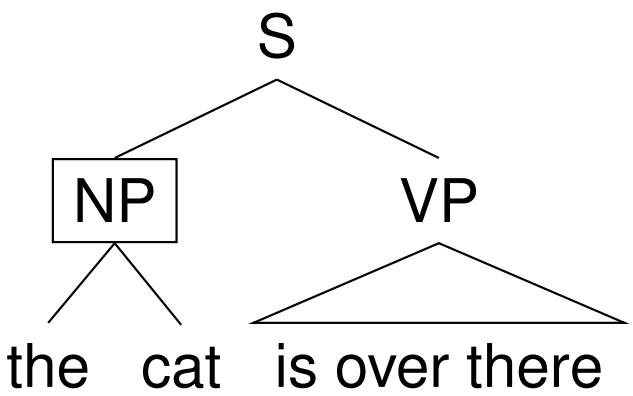
$N \to aPQ$

$\alpha N \beta \Rightarrow \alpha aPQ \beta$

$S \stackrel{*}{\Rightarrow} abcaabb.$

Semantics of Context-free languages II

Trees: (Thatcher, 1967)



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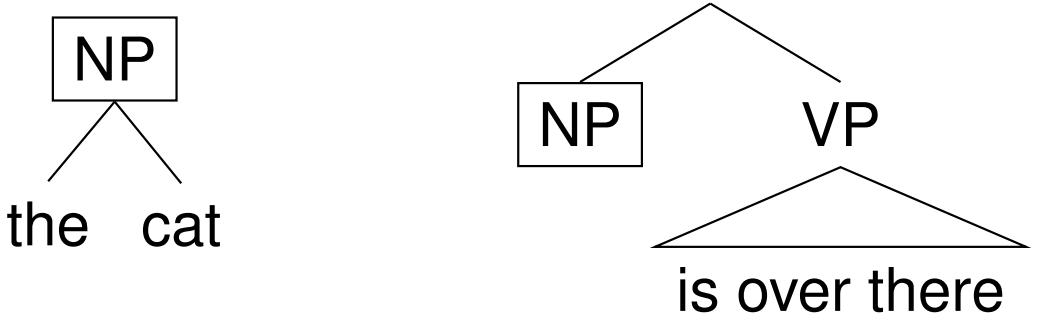

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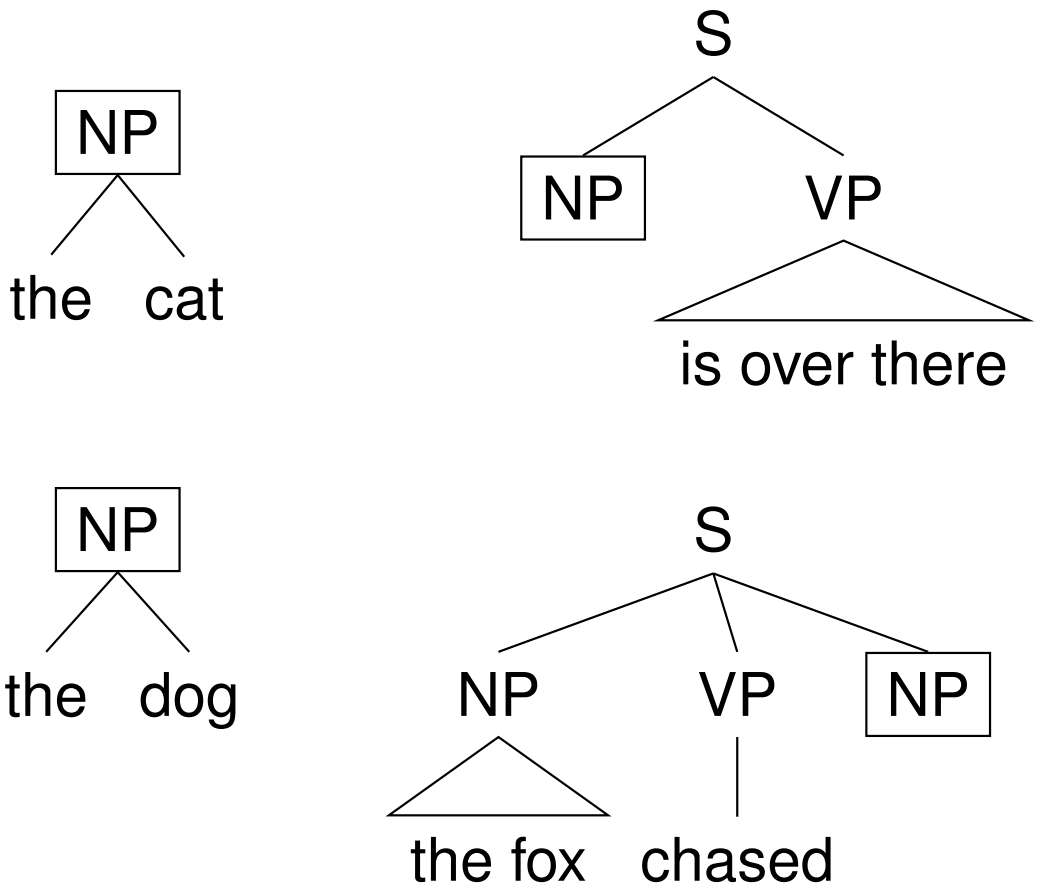

Definition

Contexts

A context is a string with a hole: $l \square r$.

Filling the hole

$l \square r \odot u = lur$

A factorisation

C is a set of contexts; S is a set of strings

$$C \odot S \subseteq L$$

Context free grammars

Contexts and yields

$\mathcal{L}(G, N) = \{w \in \Sigma^* \mid S \overset{*}{\Rightarrow} w\}$
$\mathcal{C}(G, N) = \{l \Box r \mid S \overset{*}{\Rightarrow} lNr\}.$

$$\mathcal{C}(G, N) \odot \mathcal{L}(G, N) \subseteq L$$

Congruential CFGs

Analogous to DFAs

- Nonterminals correspond to equivalence classes of strings that occur in the same set of contexts: the syntactic congruence.

Definition

Define $u^{\triangleright} = \{l \square r \mid lur \in L\}$
$u \equiv_L v$ iff $u^{\triangleright} = v^{\triangleright}$.
$[u]_L = \{v \mid u^{\triangleright} = v^{\triangleright}\}$

This gives us grammars with a “bottom-up” determinism.

Congruential CFGs

- If the language is not regular there will be an infinite number of congruence classes, but we don't need all the classes.
- Not all context-free languages can be represented with a congruential CFG: e.g. the palindrome language $\{w \mid w = w^R\}$ has congruence classes which are just single strings.
- These grammars can be efficiently learned in the exact query model.
- Like DFAs, parsing speed does not depend on the size of the grammar.
- Closely related to the NTS languages, which can be parsed in linear time.

Semantics of Context-free languages III

Equations: (Ginsburg and Rice, 1962)

Ginsburg and Rice, 1962

$S \to ab, S \to aSb, S \to \epsilon$

Interpret this as a set of equations:

$S = (a \circ b) \cup (a \circ S \circ b) \cup \epsilon$

The language is the smallest solution to these equations.

The variables take values in $\mathcal{P}(\Sigma^*)$.

Semirings

What is $\mathcal{P}(\Sigma^*)$?

Free structure over Σ with

- One associative operation $\circ$
- A unit λ
- A countable union operation $\vee$ that is associative and idempotent
- A unit with respect to $\vee$
- $\vee$ distributes over $\circ$, ...

Semirings

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- $\vee$ distributes over $\circ$, ...

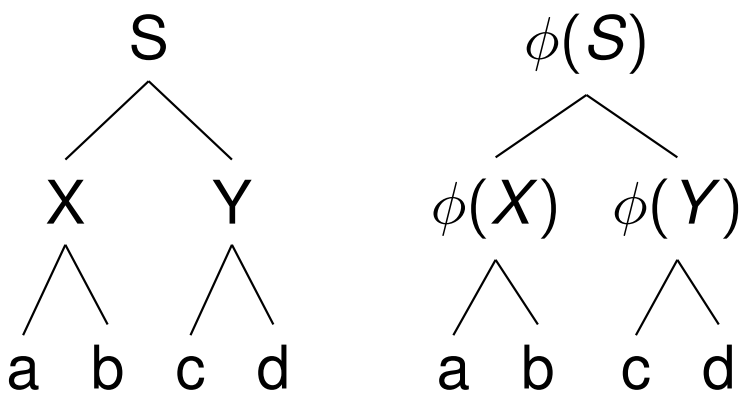
Algebraic structure

Complete idempotent semiring (CIS)

CFG-Morphism

Definition of a grammar morphism

- Suppose we have a CFG $G = \langle \Sigma, V, S, P \rangle$ and a function $\phi : V \to X$.
- Then $\phi(G)$ is the grammar $\langle \Sigma, \phi(V), \phi(S), \phi(P) \rangle$
- If $(N \to aXY) \in P$ then $(\phi(N) \to a\phi(X)\phi(Y)) \in \phi(P)$ so $\mathcal{L}(G) \subseteq \mathcal{L}(\phi(G))$



Merging

S

C

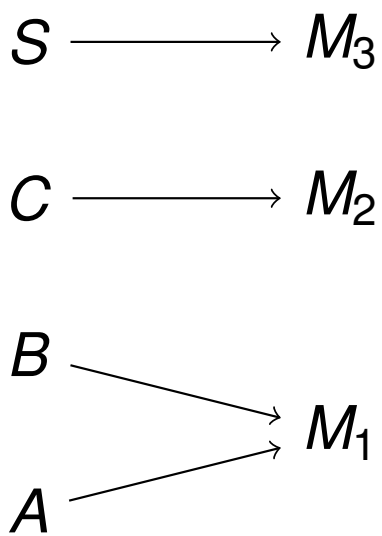
B

A

Grammar

$S \to ASB$, $S \to C$, $A \to a$, $B \to b$, $C \to c$

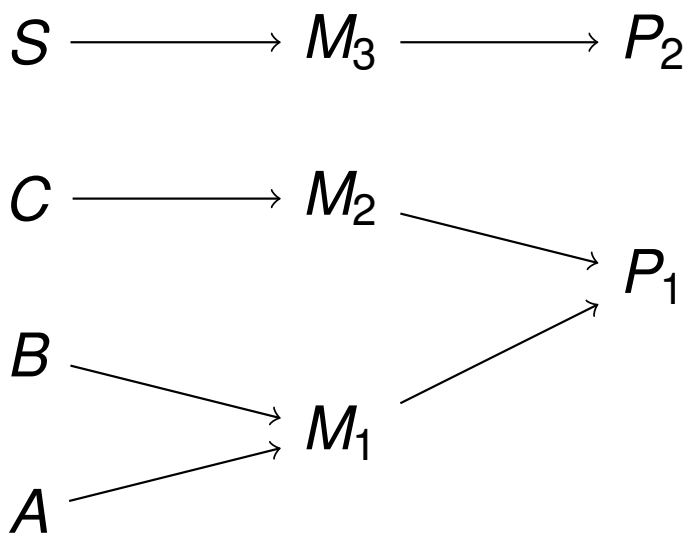
Merging



Merged grammar

$M_3 \to M_1 M_3 M_1$, $M_3 \to M_1$, $M_1 \to a$, $M_1 \to b$, $M_2 \to c$

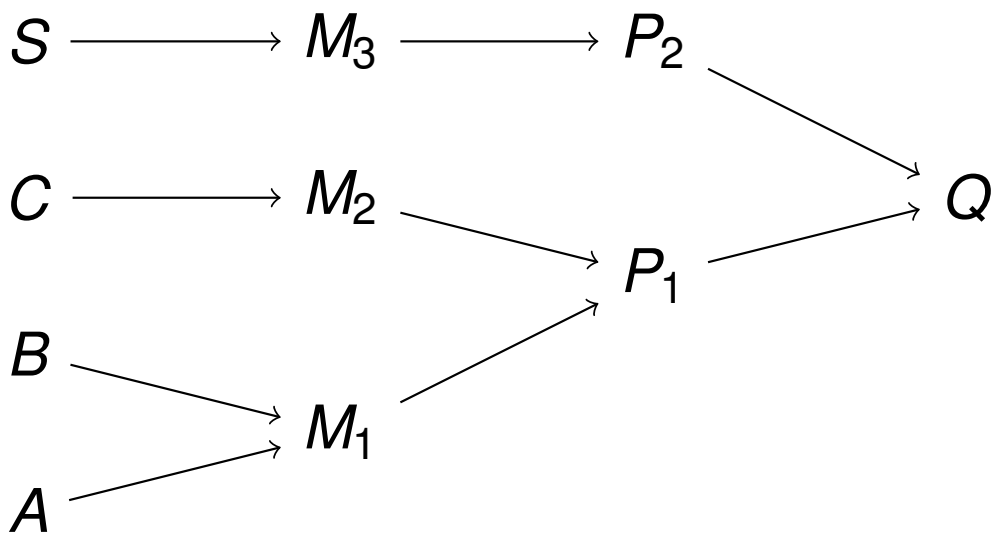
Merging



Merged grammar

$P_2 \to P_1 P_2 P_1$, $P_2 \to P_1$, $P_1 \to a$, $P_1 \to b$, $P_2 \to c$

Merging



Merged grammar

$Q \to QQQ$, $Q \to Q$, $Q \to a$, $Q \to b$, $Q \to c$

Mergable nonterminals

Suppose we have two distinct nonterminals that generate the same language:

- A, B such that $\mathcal{L}(G, A) = \mathcal{L}(G, B) = X$.
- Then we can merge them with a map $\phi(A) = \phi(B)$.
- $\phi : N \to \mathcal{L}(G, N)$

Clearly $\mathcal{L}(\phi(G)) = \mathcal{L}(G)$

Homomorphisms give morphisms

Idea

A CIS-homomorphism $f : \mathcal{P}(\Sigma^*) \to A$ gives a CFG-morphism.
$\phi_f(N) = f(\mathcal{L}(G, N))$.
Note that

- $S = (a \circ b) \vee (a \circ S \circ b) \vee \epsilon$
- So $f(S) = (f(a) \circ f(b)) \vee (f(a) \circ f(S) \circ f(b)) \vee f(\epsilon)$

Two extremes

- Identity map just merges nonterminals that generate the same language
- Map to the trivial one-element CIS merges all nonterminals into one.
  This will usually increase the language generated to Σ^*.

Pause

The central idea of the algebraic approach

- We define an algebraic structure – a string concatenation algebra appropriate for our grammar.
- We think of grammars as sets of equations in this algebra.
- Then homomorphisms of the algebra will give morphisms of the grammar.

Recognising

Residual

Define an "inverse" (actually a residual) of f, called f^*.

- $f : A \to B$ is a CIS-homomorphism
- $f^* : B \to A$ is $f(y) = \bigvee \{x \mid f(x) \leq y\}$

Definition

We say that a CIS B recognizes L if there is a surjective morphism h from $\mathcal{P}(\Sigma^*) \to B$ such that $h^*(h(L)) = L$.

Analogue of the classic automata theoretic idea of a monoid recognizing a language

There is a unique answer

We want to merge as much as possible while still generating the same language.

Theorem

Let G be a CFG over Σ and h a homomorphism $h : \mathcal{P}(\Sigma^*) \to B$. Then

- $\phi_h(G)$ defines the same language as G iff
- B recognizes L through h

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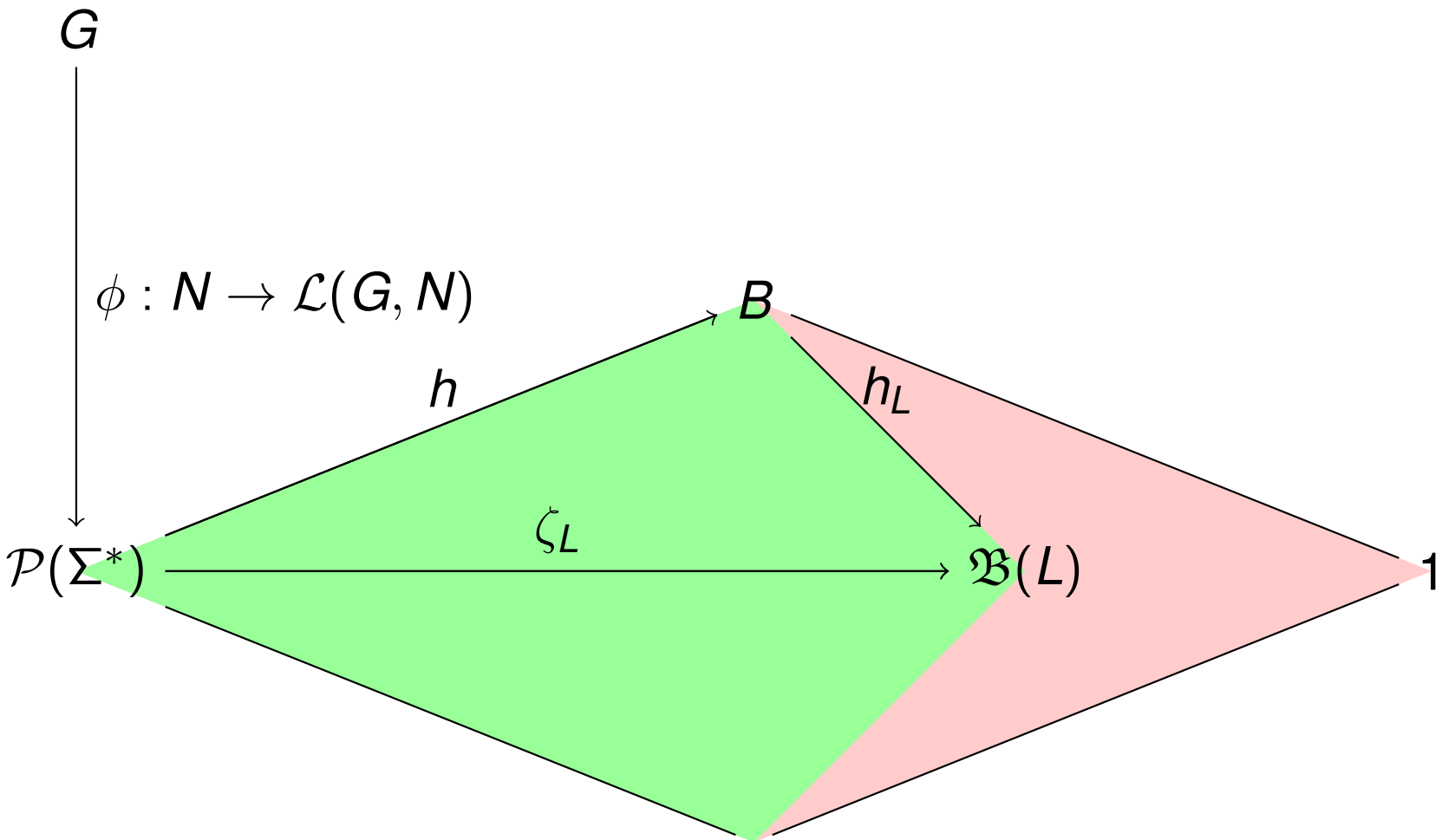
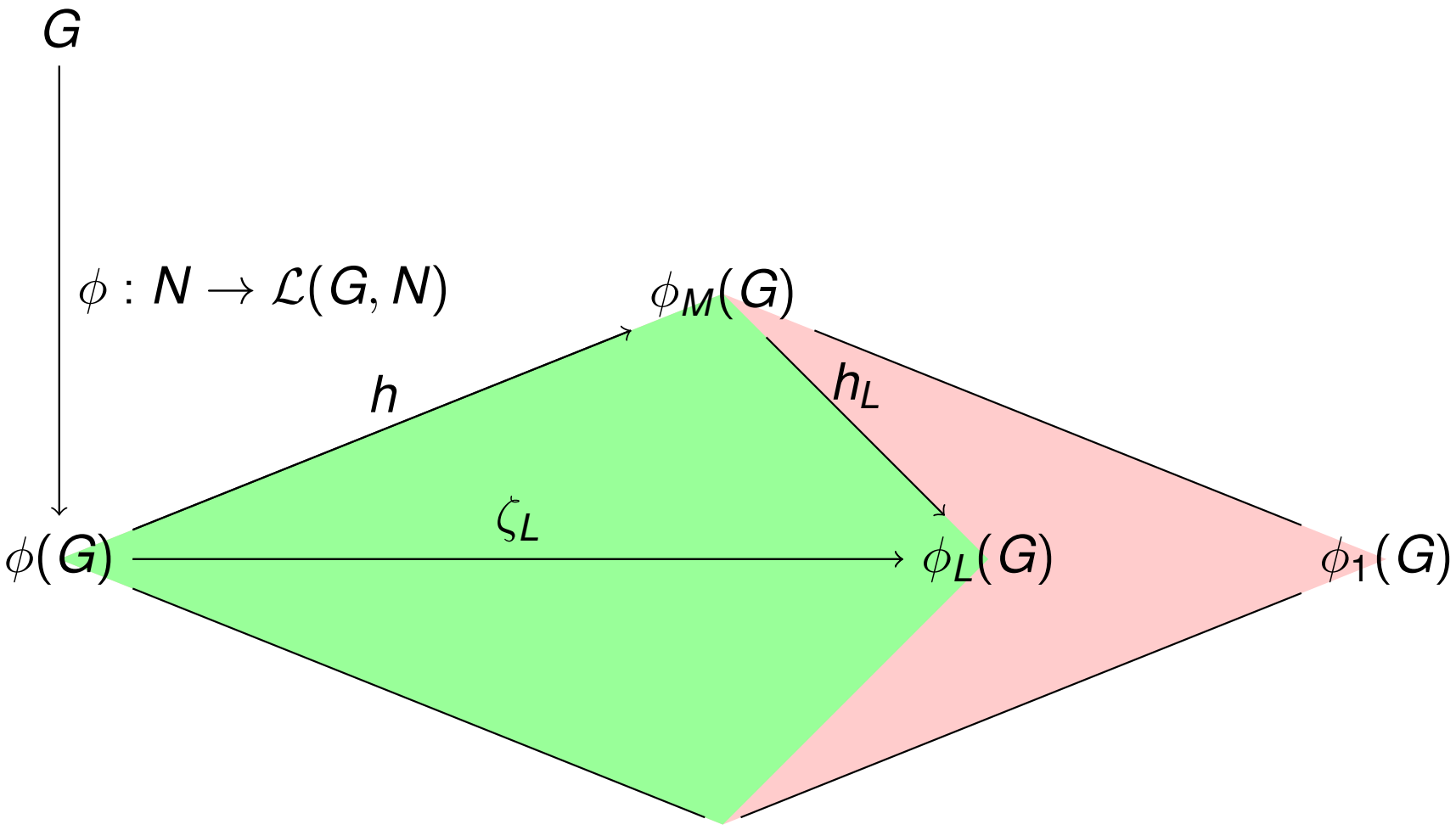
Uniqueness

There is a unique 'smallest' CIS that recognizes L $\mathfrak{B}(L)$.

The universal morphism



The universal CFG-morphism



Outline

Maximal decompositions into contexts and substrings

Definition

A *syntactic concept* is a pair $\langle S, C \rangle$ where

- where S is a set of strings, and C is a set of contexts
- $C \odot S \subseteq L$
- C and S are both maximal

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The unique smallest CIS that recognizes L is called the syntactic concept lattice (Clark, 2010), denoted $\mathfrak{B}(L)$, whose elements are these maximal decompositions.

The syntactic concept lattice

If S is a set of strings:

$$S^{\triangleright} = \{l \square r \mid \forall u \in S, lur \in L\} \tag{1}$$

If C is a set of contexts:

$$C^{\triangleleft} = \{u \in \Sigma^* \mid \forall l \square r \in C, lur \in L\} \tag{2}$$

Closed sets of strings

definition

A set of strings is closed if $X = X^{\triangleright\triangleleft}$.
$X^{\triangleright\triangleleft}$ is always closed.

Nonterminals will correspond to closed sets of strings.

Example

Suppose we have a single word $\{w\}$.
$\{w\}^{\triangleright\triangleleft}$ is the set of all words that can occur anywhere w can.

Implications

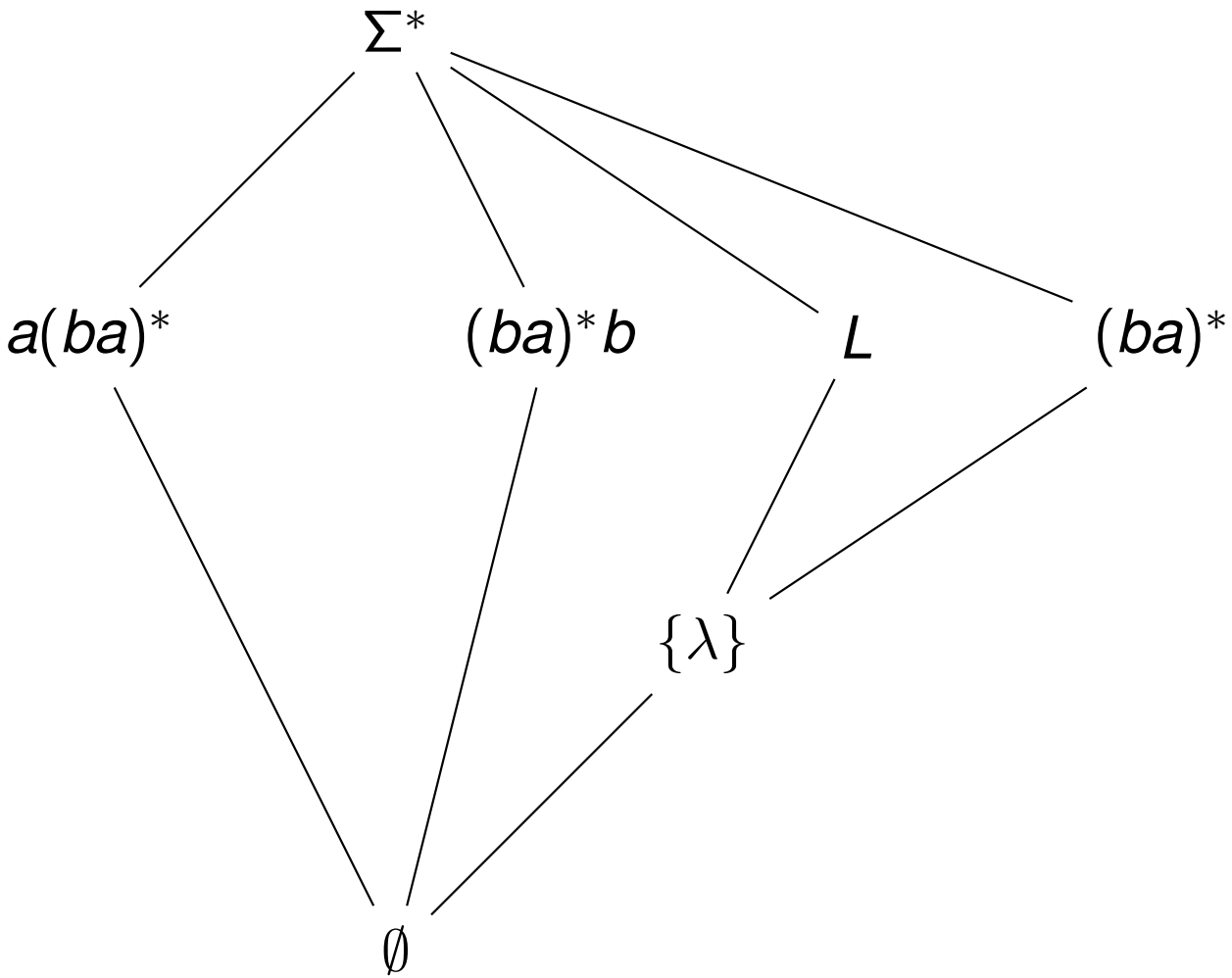
- We have a unique smallest CIS that recognises the language.
- This doesn't mean that we have a unique smallest CFG.
- But we do have a unique set of nonterminals that are now not arbitrary symbols but elements of a nontrivial algebraic structure.

The main point of this talk

Grammars without mergeable nonterminals will have nonterminals that correspond to closed sets of strings.

Simple Example

Suppose L is just regular: $(ab)^*$.



Linear Algebra 101

Dimensionality reduction

Linear map

$f : \mathbb{R}^n \to \mathbb{R}^k$, $k < n$

$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$ and $f(\alpha\vec{x}) = \alpha f(\vec{x})$

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Linear maps are just vector space homomorphisms

- $\vec{x} \cong_f \vec{y}$ iff $f(\vec{x}) = f(\vec{y})$ iff $\vec{x} - \vec{y} \in \text{Ker}(f)$
- $\mathbb{R}^n / \cong_f$ is isomorphic to $f(\mathbb{R}^n)$ which is $\mathbb{R}^k$ if it is surjective.

But if k is too small, you lose information ...

Syntactic concept lattice

Alternative definition

$X \cong_L Y$ if $X^{\triangleright} = Y^{\triangleright}$

This is a CIS-congruence and $\mathfrak{B}(L)$ is just $\mathcal{P}(\Sigma^*)/\cong_L$

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Subspaces analogy

We are just modelling the smallest subspace that captures the phenomenon we are interested in.

But using as our algebra a CIS rather than a vector space.

Lambek, 1958

"We shall assign type n to all expressions which can occur in any context in which all proper names can occur."

- Let N be the set of proper names (not necessarily closed)
- $N^{\triangleright}$ is then the set of contexts that all proper nouns can occur in.
- $N^{\triangleright\triangleleft}$ is the set of all strings which can occur in any of $N^{\triangleright}$
- n is assigned to $N^{\triangleright\triangleleft}$ which is a concept.

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- Types in Lambek grammars were intended to be closed sets of strings.
- SCLs form a sound and complete semantics for Lambek grammars (Wurm, 2012)

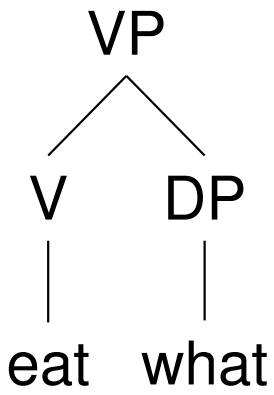
Outline

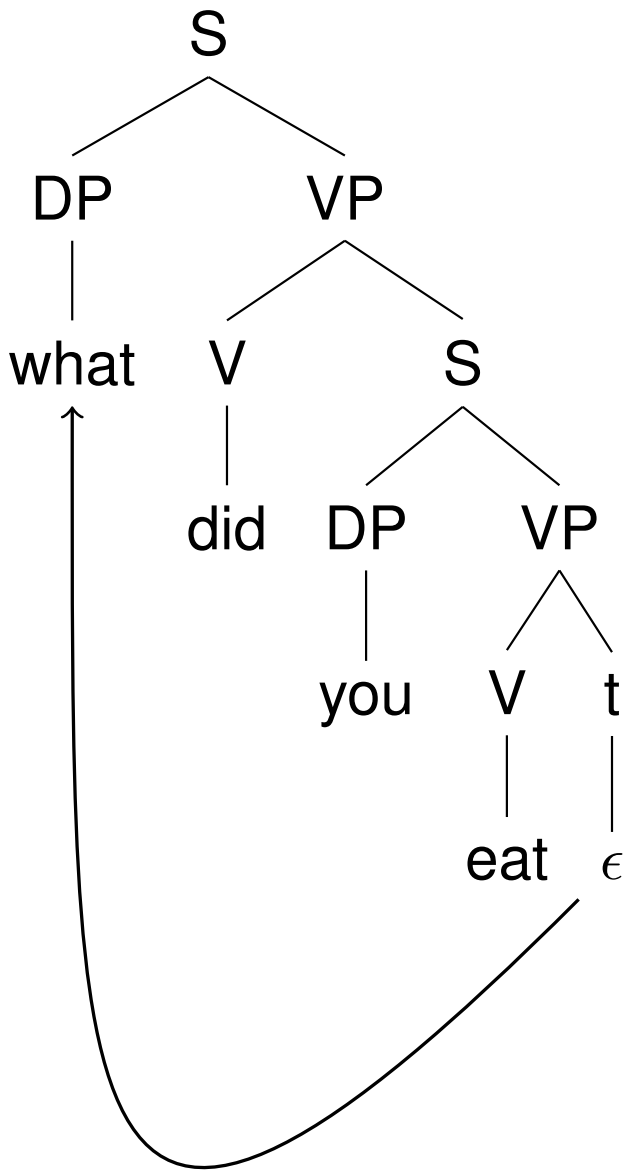
Multiple context free grammars

Mildly context-sensitive grammar formalisms

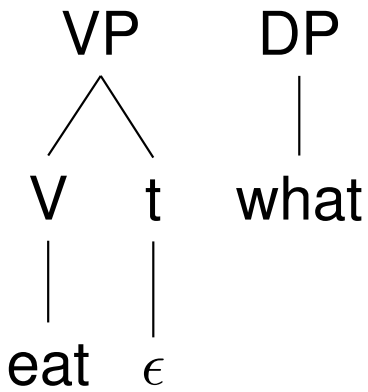
- Generalisation of context-free grammars where some nonterminals can generate tuples of strings.
- MCFGs of dimension 2 have nonterminals which generate pairs of strings: $N \stackrel{*}{\Rightarrow} \langle u, v \rangle$
- Essential for cross-serial dependencies in Dutch, Swiss German etc.
- Useful for modeling displacement in language with overt movement (English)

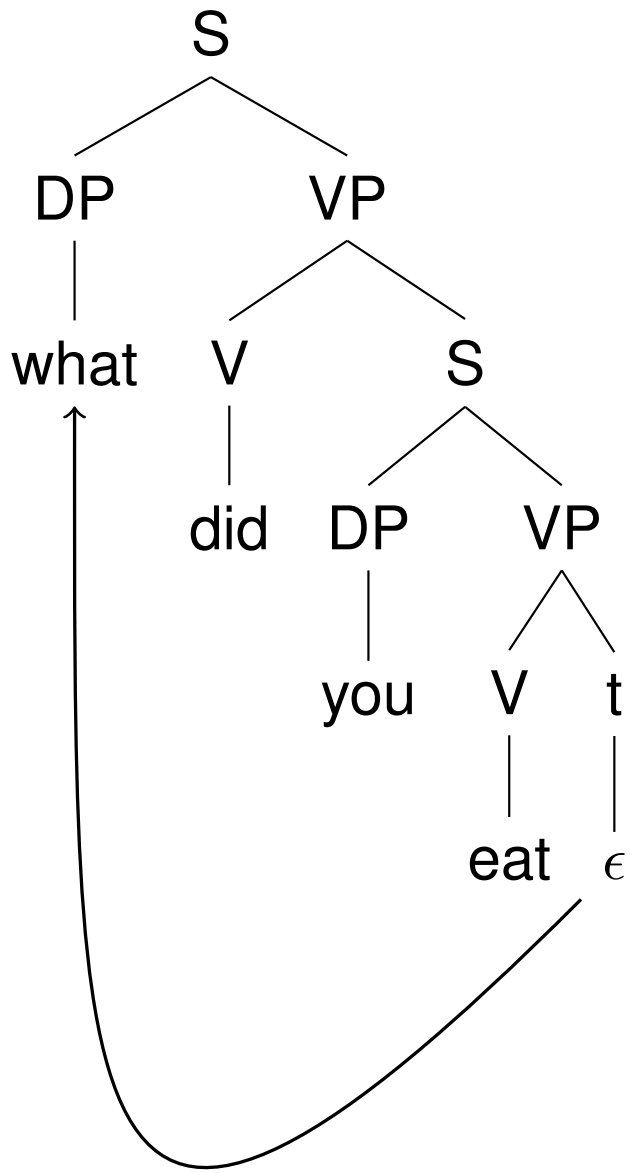
Movement





Movement





Syntactic concept lattice of order 2

2-contexts and 2-words

$l\Box m\Box r \odot \langle u, v \rangle = lumvr$

Given a set of pairs of strings X define

$$X^{\blacktriangleright} = \{l\Box m\Box r \mid l\Box m\Box r \odot X \subseteq L\}$$

Given a set of 2-contexts C we define

$$C^{\blacktriangleleft} = \{(u, v) \mid C \odot (u, v) \subseteq L\}$$

X is closed iff $X = X^{\blacktriangleright\blacktriangleleft}$.

Syntactic example

- Which book did Mary read?
- Which cake did Mary eat?

$(\square \text{ did Mary } \square \text{ ? }) \odot \langle \text{ which book, read} \rangle$
$(\square \text{ did Mary } \square \text{ ? }) \odot \langle \text{ which cake, eat} \rangle$

Minimality result

For a given MCFL_2, L, the minimal MCFG_2 will have nonterminals where:

- The nonterminals of dimension 1 will correspond to closed sets of strings.
- The nonterminals of dimension 2 will correspond to closed sets of pairs of strings.

Outline

Answers

What are the elements of $V = \{N, P, Q \dots\}$?

They are elements of the syntactic concept lattice.

What does S mean?

S is the concept/closed set of strings L.

What does a production like $N \to aPQ$ mean?

It refers to an inclusion relation $N \supseteq \mathcal{C}(a) \circ P \circ Q$

Syntactic structure and Strong learning

Hard question

We know that minimal (M)CFGs are *weakly* adequate for all (M)CFLs
Are the minimal grammars strongly adequate for natural language syntax?

Implications of using minimal grammars

- If two words *a* and *b* are distributionally identical then they must be syntactically identical.
- If a sentence *lar* has *n* distinct parses, then *lbr* must have *n* distinct parses of the same form.
- Very strong prediction that seems to be true, and is not made by any other theory.

Conclusion

- The syntactic concept lattice is a canonical object that provides a denotation for context free grammars; with a rich algebaric structure.
- The elements of it are maximal factorisations of the language into sets of contexts and sets of strings.
- We can compute (fragments of it) from knowledge of the strings in the language, which gives algorithms for weak and strong learning of CFGs.
- We can extend it to MCFGs;
- More generally we can view grammars algebraically; homomorphisms of the algebra give morphisms of the grammars.

Thanks

Thanks to . . .

Ryo Yoshinaka, Hans Leiss, Jason Eisner, . . .

References

“The syntactic concept lattice”, to appear (2013?), Journal of Logic and Computation.