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A note about Norbert Wiener and his contribution to Harmonic Analysis and Tauberian Theorems

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Abstract. In this note we explain the main motivations Norbert Wiener had for the creation of his Generalized Harmonic Analysis [13] and his Tauberian Theorems [14]. Although these papers belong to the most pure mathematical tradition, they were deeply based on some Engineering and Physics Problems and Wiener was able to use them for such diverse areas as Optics, Brownian motion, Filter Theory, Prediction Theory and Cybernetics.

Keywords: Norbert Wiener, Harmonic Analysis, Tauberian Theorems, History of Science
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1. INTRODUCTION

Norbert Wiener (1894-1964) probably was, jointly with G. D. Birkhoff, one of the first mathematicians in USA who got international prestige. He was a prodigy who graduated from Tufts College (i.e., the actual Tufts University) at fourteen and then started in Philosophy of Mathematics, defending a doctoral dissertation in this field at the age of 19 at Harvard's Philosophy Department. Then he got a grant to make some postdoctoral research with B. Russell at Cambridge (England) and, after some bad experiences with Russell, he turned from Philosophy to Mathematical Analysis. Thus, under the influence of Hardy, he started a brilliant career as a mathematician with the introduction of a mathematical theory for the Brownian motion between 1921 and 1923 [7],[8],[9] (with this work he became an initiator on the study of Stochastic Processes and a precursor of Probability Theory in infinite dimensional spaces). The next three steps given by Wiener in the mathematical world were the solution of Zaremba's problem [10], [11], the creation of Generalized Harmonic Analysis (GHA) [13] and the development of a deep theory motivated by Tauberian theorems [14].

Curiously, his mathematical work, although it got the highest level of abstraction, was always motivated by Physics and/or Engineering. In particular, his GHA was a deep tool for the study of weak currents, a subject whose importance has been an increasing function of time in the 20th century. Concretely, GHA was a main tool for the treatment of many Engineering problems -such as filtering (i.e. the separation of noise from message) and prediction theory- and Wiener would use it also for the creation of Cybernetics.

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In this note we will explain the motivations he had for the creation of GHA and we will also say something about its applications.

2. FROM HEAVISIDE'S OPERATIONAL CALCULUS TO DISTRIBUTIONS VIA HARMONIC ANALYSIS

Although several biographies of Wiener have been published, we think that none of them gives the importance it deserves to a paper published by Wiener in 1926 with the intention to give a solid mathematical basis to Heaviside's Operational Calculus [12]. It is because of this that we would like to stand out this paper in this short section.

Wiener entered at MIT in 1919 and it was soon assumed that he should be the man who would help the Electrical Engineering Department's people to put several mathematical tools they used frequently in a sound foot. In particular, the most urgent task they assigned him was to create a solid mathematical foundation for Heaviside's Operational Calculus (HOC). This question was proposed by D.C. Jackson -who was in 1920 the Chair of the department and had been a friend of Wiener in his childhood. The main idea of HOC is to treat the differential operator $p=\frac{d}{dt}$ as an algebraic object, included in (an unknown) field, and to identify its algebraic inverse $q=p^{-1}$ with the integral operator $q\{f\}(x)=\int_0^x f(t)dt$. With these assumptions, HOC would be useful for the solution of many differential equations by the procedure of shifting the differential problem to an algebraic problem. The best form to understand this question is just to solve an easy example. So, let us assume that we are interested in the solution of an RC-circuit, which is described by the ODE $RCy'(t)+y(t)=x(t)$, where R,C are constants -the resistance and the conductance-, $x(t)=i(t)u(t)$ is the potential difference introduced in the system (here $u(t)$ represents the Heaviside unit step function, which is equal to 0 for $t\leq 0$ and 1 for $t>0$) and $y(t)$ is the output of the system. Let us consider the special case $x(t)=Eu(t)$, where E is a given constant (this is an important case which models the problem of introducing at $t=0$ a constant difference of potential E into the system) Any practitioner of HOC, would operate in the following way: firstly, he rewrites the problem as $(RCp+1)y(t)=x(t)$, so that $y(t)=\frac{1}{RCp+1}x(t)$, then he develops the quotient $\frac{1}{RCp+1}$ as a power series of $q=\frac{1}{p}$,

$$\frac{1}{RCp+1}=(-1)\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(RC)^{k+1}}q^{k+1} \tag{2.1}$$

and finally, he uses that $q^k\{u\}(t)=\frac{t^k}{k!}u(t)$ for $k=1,2,\cdots,$, so that

$$y(t)=(-1)E\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(RC)^{k+1}}\frac{t^{k+1}}{(k+1)!}u(t)=E(1-e^{\frac{-t}{RC}})u(t),$$

which is the exact solution of the equation. Of course, there are many steps where this method is not rigorous. For example, the meaning of $q=\frac{1}{p}$ is not clear and it is also unclear in which sense the power series given by (2.1) converges. So, we should wonder why this method produces a correct answer!

Wiener was asked to give an explanation of HOC that should satisfy both to engineers and mathematicians. In his 1926 paper, Wiener exposed the question with the following words:

The problem of obtaining a rigorous interpretation of Heaviside's Operational Calculus (...) is still open. There are several paths which seem to lead to this goal. Besides the Volterra theory of permutable kernels and the Pincherle theory of transformations of power series, the Laplace transformation and the Fourier integral appear to be promising tools. Like the theory of Pincherle, however, the theory of Laplace transformation is directly applicable only to analytic functions. The Fourier integral, which may be regarded as derived from a complex form of the Laplace transformation, is not open to this objection. On the other hand, the functions to which the classical form of the Fourier integral applies are subjected to very severe restrictions as to their behavior at infinity.

But he was convinced that Fourier analysis would be the key for a solution of the problem, been a main reason for this his observation that HOC makes an extensive use of the operators of the form $f(\frac{d}{dt}) := \sum_{k=0}^{\infty} a_k \frac{d^k}{dt^k}$, where $f(t) = \sum_{k=0}^{\infty} a_k t^k$ is a power series, and these operators satisfy the following

Lemma 2.1. *Take* $g(t) = e^{\mathbf{i}wt}$ *and* $f(t) = e^{\mathbf{i}\alpha t}$. *Then* $L = g(\frac{d}{dt})$ *satisfies:*

$$L(f) = f(\mathbf{i}w)f(t) = g(\mathbf{i}\alpha)f(t)$$

Proof. By definition, $L = g(\frac{d}{dt}) = \sum_{k=0}^{\infty} \frac{(\mathbf{i}w)^k}{k!} \frac{d^k}{dt^k}$, so that

$$\begin{aligned} L(f) &= \left[\sum_{k=0}^{\infty} \frac{(\mathbf{i}w)^k}{k!} (\mathbf{i}\alpha)^k\right] f(t) = g(\mathbf{i}\alpha)f(t) \\ &= \left[\sum_{k=0}^{\infty} \frac{(\mathbf{i}\alpha)^k}{k!} (\mathbf{i}w)^k\right] f(t) \\ &= f(\mathbf{i}w)f(t) \end{aligned}$$

He resumed this property with the following words:

When applied to the function e^{nit}, the operator $f(d/dt)$ is equivalent to the multiplier $f(ni)$. The result of applying a given operator to a given Fourier integral may naturally be conceived, then, as the multiplication of each e^{nit} term in the integral by a multiplier depending only on n. That is, the operator d/dt has no particular location in the complex plane, but ranges up and down the entire axis of imaginaries. It is thus too much to expect in general that any particular series expansion or other analytic representation of f should make $f(d/dt)$ converge when applied to a purely arbitrary function.

In the present paper, the device is adopted of dissecting a function into a finite or infinite number of ranges of frequency, and of applying to each range the particular expansion of an operator yielding results convergent over that range.

It is by this method of dissection that Heaviside's asymptotic series are justified.

These ideas were a strong motivation for Wiener to construct a Generalized Harmonic Analysis (GHA) which should be applicable to quite general functions. Particularly, to aperiodic functions not decaying in the infinite. Another main motivation that would be stood out by Wiener a few years later, with the publication of his seminal paper on GHA was the existence of many physical processes where classical Harmonic Analysis is not helpful:

The two theories of Harmonic Analysis embodied in the classical Fourier series and the theory of Plancherel do not exhaust the possibilities of Harmonic Analysis . The Fourier series is restricted to the very special class of periodic functions, while the Plancherel theory is restricted to functions which are quadratically summable, and hence tend on the average to zero as their argument tends to infinity. Neither is adequate for the treatment of a ray of white light which is supposed to endure for an indefinite time. Nevertheless, the physicists who first were faced with the problem of analyzing white light into its components had to employ one or other of these tools...

Moreover, in his 1926 paper Wiener also initiated a study of operators, since they were a basic tool for HOC. This led him to introduce a theory for causal (or, in his original notation, retrospective) operators that would produce some interesting results. The operator L is "causal" if (and only if) for every real number t we have that $f_{|(-\infty,t]} = g_{|(-\infty,t]}$ implies that $L(f)(t) = L(g)(t)$. These operators are very important in Engineering since they model physically realizable systems. Wiener investigated them with insistence in the 1920's and the 1930's and, finally, he was able to prove several fundamental results about them. In particular, he proved with his Ph.D. student Y. W. Lee (see [4]) that every physically realizable system is in fact realizable in the metal (this means that they were able to construct an electrical net -the Lee-Wiener net- that approximates any physically realizable system with arbitrary precision[1]) Another fundamental result he got with the help of the English mathematician Paley was the mathematical characterization of physically realizable systems in the frequency domain as those of the form $L(X)(\xi) = X(\xi)H(\xi)$ for which $H(\xi) \in L^2(\mathbf{R})$ and

$$\int_{-\infty}^{\infty} \frac{|\log|H(\xi)||}{1+\xi^2} d\xi < \infty.$$

This was a deep result and Wiener was so proud of it that he mentioned it several times in his mathematical and biographical writings. For example, in [15,p. 37] he said:

This [result] plays a very important part in the theory of filters. It states that, in any realizable network whatever, the attenuation, taken as a function of the frequency ω, and divided by $1+\omega^2$, yields an absolutely integrable function of the frequency. This results from the fact that the attenuation is the logarithm of the absolute value of the transform of [the response to the unit impulse] $f(t)$ which vanishes for negative t; or, in other words, because strictly no network can foretell the future. Thus no filter can have infinite attenuation in any finite band. The perfect filter is physically unrealizable by its very nature, not merely because of the paucity of means at our disposal. No instrument acting solely on the past has a sufficiently sharp discrimination to separate one frequency from another with unfailing accuracy.

Moreover, in his autobiography [16,p. 168], when speaking about his mathematical work with Paley, he said:

One interesting problem which we attacked together was that of the conditions restricting the Fourier transform of a function vanishing on the half line. This is a sound mathematical problem on its own merits, and Paley attacked it with vigor, but what helped me and did not help Paley was that it is essentially a problem in electrical engineering. It had been known for many years that there is a certain limitation on the sharpness with which an electric wave filter cuts a frequency band off, but the physicists and engineers

[1] They created a patent in USA for this net and later on they sold the rights to the AT&T company [17]

had been quite unaware of the deep mathematical grounds for these limitation. In solving what was for Paley a beautiful and difficult chess problem, completely contained within itself, I showed at the same time that the limitations under which the electrical engineers were working were precisely those which prevent the future from influencing the past.

Since the main motivation for HOC was in applications, Wiener decided to use his method to solve the most important equation of that time for an electrical engineer: the telegraph equation

$$v_{xx} = RCv_t + LCv_{tt}; v(x,0) = 0, v(0,t) = f(t).$$

Here, $v(x,t)$ represents the voltage at a point in a cable which is at distance x of the origin (the point where we introduce the voltage) and at time t. Thus $f(t) = v(0,t)$ represents the voltage we are introducing at an end (the origin) of the cable at time t and we are interested in the quantity $v(L,t)$, where L represents the length of our cable.

Now, let us think a little bit about this equation. Obviously, the input $f(t)$ for a typical telegraphic message will be a discontinuous function. This implies, in particular, that $v(0,\cdot)$ is discontinuous, so that the meaning of the derivatives appearing in the equation is by no way clear. This led Wiener to claim that

(...) there are cases where v must be regarded as a solution of our differential equation in a general sense without possessing all the orders of the derivative indicated in the equation and indeed without been differentiable at all. It is a matter of some interest, therefore, to render precise the manner in which a non-differentiable function may satisfy in a generalized sense a differential equation.

Of course, this would be nothing but good desires if Wiener had not made his own contribution by defining what he meant for a “generalized solution":

(...) Let $G(x,y)$ be a positive function which is infinitely differentiable inside a certain polygonal bounded region R of the XY-plane with the property that G and all its derivatives vanish on ∂R and over the exterior of R. Then there is a function $G_1(x,y)$ such that

$$\int\int_R (Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu)G(x,y)dxdy =$$
$$= \int\int_R u(x,y)G_1(x,y)dxdy$$

for all function u having integrable derivatives of the first orders, as can be proved by integrating by parts. Thus, a necessary and sufficient condition for u to verify the differential equation $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$ almost everywhere is that

$$\int\int_R u(x,y)G_1(x,y)dxdy = 0$$

for every function $G_1(x,y)$ -since the functions G form a complete system on any region and u has the required derivatives. We can, hence, treat the functions u been orthogonal to all functions G_1 as solutions of the differential equation in a generalized sense.

Thus, the 1926 paper was also important because in it Wiener introduced a concept of “generalized solution” of a P.D.E. that, in modern terminology is exactly the same as our weak solutions of P.D.E.s, so that we can claim that Wiener introduced the concept of distribution (in the Schwartz’s sense) with two decades in advance!

3. GENERALIZED HARMONIC ANALYSIS AND TAUBERIAN THEOREMS

Motivated by his contribution to HOC and by some physical phenomenons like the the white light, Wiener devoted many years to the creation of an Harmonic Analysis valid for a large class of functions. The concrete set of functions he decided to study was the so called Wiener class **S** whose elements are functions $f:\mathbf{R}\to\mathbf{C}$ which are Lebesgue measurable and whose covariance function

$$\phi(t)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t+s)\overline{f(s)}ds$$

is a well defined number for every $t\in\mathbf{R}$ and $\phi\in C(\mathbf{R})$. This set is large enough to contain the main physical (and mathematical) functions on which Wiener was interested, including white light, the Bohr-Besicovitch class of quasi-periodic functions and the sample functions associated to many stochastic processes (including Wiener Motion!).

From a mathematical point of view the best way to motivate the study of covariance functions is assuming that $f(t)$ is a trigonometric polynomial $f(t)=\sum_{k=0}^{n}a_ke^{i\lambda_kt}$. In such a case the associated covariance function $\phi(t)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t+s)\overline{f(s)}ds$ is easily computed as

$$\phi(t)=\sum_{k=0}^{n}|a_k|^2e^{i\lambda_kt}.$$

This means that the covariance function $\phi(t)$ preserves the information of $f(t)$ related to the amplitude (but eliminates the phases) of its spectrum. In particular, it preserves all information about the energy of the signal.

The covariance function has many interesting properties. For example, it satisfies the inequality $|\phi(t)-\phi(t+\varepsilon)|\leq\sqrt{[2\phi(0)-2\mathbf{Re}(\phi(\varepsilon))]\phi(0)}$. This implies that $\phi(t)$ is continuous at the origin if and only if it is continuous at every point of the real line, since $\phi(0)$ is a real number. Moreover, $\phi(0)=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|f(s)|^2ds$ can be interpreted as the power of the signal $f(t)$.

Wiener noted that if one tries to identify which part of the power of $f(t)$ is concentrated between the frequencies $-A$ and A and takes the limit $A\to+\infty$, the result of that limit is precisely $\lim_{\varepsilon\to0}\phi(\varepsilon)$ and this limit is always less or equal to $\phi(0)$. Obviously, this motivated him to restrict the attention to those functions $f(t)$ whose covariance function is continuous. (Otherwise, some hidden frequencies would be necessary for the creation of an Harmonic Analysis applicable to the function $f(t)$) In order to introduce his Harmonic Analysis , Wiener needed to define a concept of spectrum for the elements of the Wiener class **S** and to construct a transform sending the elements of **S** to their spectrum. Moreover, this transform should be invertible and preserve the power of the signals. A first step to get his transform was to prove that the elements of the Wiener class satisfy the inequality $\int_{-\infty}^{\infty}\frac{|f(t)|^2}{1+t^2}dt<\infty$ which guarantees the existence of the limit (in the sense of the energy norm):

$$\mathscr{W}(f)(\xi)=\lim_{A\to+\infty}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]\frac{f(t)e^{-i\xi t}}{-it}dt+\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\frac{f(t)(1-e^{-i\xi t})}{it}dt.$$

This function $\mathscr{W}(f)$ is very important since it is possible to prove that if $f(t)$ is a function with finite energy and $F(\xi)$ is his Fourier transform, then $\mathscr{W}(f)$ is a primitive of $F(\xi)$. On the other hand, Wiener proved that

$$\phi(t) = \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\mathscr{W}(f)(w+\varepsilon)-\mathscr{W}(f)(w-\varepsilon)|^2 e^{iwt}dw$$

(here the limit is also in the sense of the energy norm) and, as a consequence of this, there exists a monotonous, decreasing non-negative and bounded variation function $\Lambda(\xi)$ such that $\phi(t) = \int_{-\infty}^{\infty} e^{it\xi}d\Lambda(\xi)$. Furthermore, $\Lambda(\xi)$ can be recovered in terms of the covariance function via de formula

$$\lim_{A\to+\infty}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]\frac{\phi(t)e^{-i\xi t}}{-it}dt+\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\frac{\phi(t)(1-e^{-i\xi t})}{it}dt = \Lambda(\xi)+cte.$$

Finally, if we impose $\lim_{\xi\to-\infty}\Lambda(\xi) = 0$ then $\Lambda(\xi)$ represents the total power included in the spectrum of the signal $f(t)$ for the frequencies between $-\infty$ and ξ. Wiener named $\Lambda(\xi)$ the integrated spectrum (or periodogram) of $f(t)$. The integrated spectrum can adopt many forms, including discrete, continuous and mixed spectrums, and Wiener devoted a lot of work to their study. It is important to note that all the results we have stated about the integrated spectrum till now were re-stated by Benedetto in the 1970's (see [3]) in terms of distributions and can be summarized just claiming that for every function $f(t)$ satisfying $\int_{-\infty}^{\infty}\frac{|f(t)|^2}{1+t^2}dt < \infty$, the Wiener's transform $\mathscr{W}(f)$ is well defined and verifies

$$\mathscr{W}(f)' = \mathscr{F}(f)$$

in the distributional sense.

Now, in order to close his theory and been able to classify it as "an Harmonic Analysis", Wiener needed to prove a result analogous to the classical Plancherel formula. This was the hardest part of his theory. Already in 1926 he had the statement of the result, which claims that the formula

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2dt = \lim_{h\to 0}\frac{1}{4\pi h}\int_{-\infty}^{\infty}|\mathscr{W}(f)(\xi+h)-\mathscr{W}(f)(\xi-h)|^2d\xi$$

holds true for every function $f(t)$ in the Wiener class **S**. Now, his proof strongly depended on the assumption that all positive functions g satisfy the identity

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}g(t)dt = \lim_{h\to 0}\frac{2}{\pi h}\int_{0}^{\infty}g(t)\frac{\sin^2(ht)}{t^2}dt, \tag{3.1}$$

a fact which is by no way easy to demonstrate. In fact, he needed a lot of time and imagination to attack this question. It was precisely the necessity to prove the identity above which led him to investigate Tauberian Theorems.

At that time the most recognized experts on Tauberian Theorems were Hardy and Littlewood. Concretely, they had a family of results about the asymptotics (for $y\to 0^+$) of many integrals of the type $\int_0^{\infty}\phi(\frac{x}{y})\varphi(x)dx$. To study these integrals, Wiener proposed the change of variables $x = e^{-s}$, $y = e^{-t}$ in conjunction with the introduction of the new functions $f(u) = \phi(e^u)$ and $g(u) = e^{-u}\varphi(e^{-u})$. This produced the following identities

$$\int_0^\infty \phi(\frac{x}{y})\varphi(x)dx = \int_{-\infty}^\infty \phi(e^{t-s})\varphi(e^{-s})e^{-s}ds = \int_{-\infty}^\infty f(t-s)g(s)ds.$$

Thus, via this change of variables Wiener transformed the study of Hardy and Littlewood's Tauberian theorems into the study of the limit $\lim_{t\to+\infty}(g*f)(t)$, a problem about convolutions or (as Wiener liked to call them) wave filters. This was good for Wiener, since he was an expert on Fourier transforms. He transformed the study of Tauberian theorems into the following one: given a wave filter $L(f) = g*f$ (i.e., given the only wave filter with unit impulse response given by $g(t)$), for which inputs $f(t)$ may we guarantee that the output $y(t) = L(f)(t)$ has a well defined limit for $t \to +\infty$? Moreover, he proved the following

Theorem 3.1 (Wiener's Great Tauberian Theorem). *Let us assume that $f \in L^\infty(\mathbf{R})$ and $g \in L^1(\mathbf{R})$. If there exists a new function $g_0 \in L^1(\mathbf{R})$ with the property that its Fourier transform $G_0 = \mathscr{F}(g_0)$ satisfies $G_0(\xi) \neq 0$ for every $\xi \in \mathbf{R}$ and there exists the limit*

$$\lim_{t\to\infty}(g_0*f)(t) = A\int_{-\infty}^\infty g_0(s)ds, \tag{3.2}$$

then the limit $\lim_{t\to\infty}(g*f)(t)$ also exists and it satisfies (with the same constant A) the identity

$$\lim_{t\to\infty}(g*f)(t) = A\int_{-\infty}^\infty g(s)ds. \tag{3.3}$$

Furthermore, if the function $g_0(t)$ satisfies $\int_{-\infty}^\infty g(s)ds \neq 0$ and for every pair of functions $f \in L^\infty(\mathbf{R})$ and $g \in L^1(\mathbf{R})$ we have that (3.2) implies (3.3), then the Fourier transform of g_0 satisfies $\mathscr{F}(g_0)(\xi) \neq 0$ for every $\xi \in \mathbf{R}$.

He also proved a version of the theorem above adapted to convolutions against measures, $(g*\eta)(t) = \int_{-\infty}^\infty g(t-s)\eta(ds)$, where we have interchanged the function $f(t)$ by a measure η defined on the real line. It is precisely the use of these measures what makes possible a bridge between the problem of estimating improper integrals and the problem of the convergence of numerical series (which is the truly origin of Tauberian theorems).

It is obvious that a proof of Wiener's Great Tauberian Theorem is a difficult task. Wiener's approach was based on the use of several technical lemmas, the most important of which was the following nice result:

Lemma 3.2 (Wiener's Small Tauberian Theorem). *Let us assume that $f(t)$ is a continuous 2π-periodic function such that $f(t) \neq 0$ for every $t \in \mathbf{R}$ and let $\{c_k(f)\}_{k=-\infty}^\infty$ denote the sequence of Fourier coefficients of f. Then $\{c_k(f)\}_{k=-\infty}^\infty \in \ell^1(\mathbf{Z})$ if and only if $\{c_k(1/f)\}_{k=-\infty}^\infty \in \ell^1(\mathbf{Z})$*

Wiener's proof of this Lemma was frontal. Basically, he directly computed the Fourier expansion of $1/f$ in terms of the Fourier coefficients of f and then he made some estimations which led him (after several delicate tricks such as the multiplication by a trapezoidal function and the use of partitions of the unity) in a very elegant way to the desired result. It is perhaps because of his method of proof that he had no conscience of the fundamental fact that his theorem can be interpreted as a characterization of the invertible elements of $\ell^1(\mathbf{Z})$ viewed as a Banach algebra with the convolution of sequences as a product. This fact was stood out a few years later by the Russian mathematician I. M. Gelfand and motivated him to create the new branch of Functional Analysis of Banach Algebras. By the way, there exists still another point of view that makes Wiener's Small Tauberian Theorem a natural result, being the main idea of this new perspective the fact that there exists a strong connection between the rate of decaying to zero of the Fourier coefficients of a function and its smoothness properties. This connection is well know to every expert on Approximation Theory and can be stated as follows: The more rapid decaying of Fourier coefficients, the more smooth the function is (and viceversa, the more smooth the function is, the more fast Fourier coefficients converge to zero). Thus Wiener's Small Tauberian Theorem can be viewed a a result claiming that functions which are smooth up to a certain degree (i.e. the smoothness associated to the fact that $\{c_k(f)\}_{k=-\infty}^{\infty} \in \ell^1(\mathbf{Z})$) have the property that their algebraic inverses are smooth up to the same degree. In fact, we are happy to say that the analogous result holds true for all concepts of smoothness. This was proved by J. M. Almira and U. Luther in [1].

The importance of Wiener's Tauberian Theorems is not limited to their originality. The main thing was that Wiener recovered all classical Tauberian theorems as corollaries of his Great Theorem. Moreover, he also was able to prove some new theorems, including a Tauberian theorem for the Lambert series whose proof had resisted the attack of many mathematicians and from which a new beautiful proof of the Prime Number Theorem was obtained. On the other hand, Wiener succeeded to prove the identity (3.1) and, as a consequence, to put in a sound foot his GHA.

As soon as his friend Tamarkin (at that period a professor at Brown University) had news from Wiener about his results, he aimed him insistently to write two big monographic papers on the subject of GHA and Tauberian Theorems, respectively. Wiener agreed and wrote both papers. The first one contained his results on GHA and was published in Acta Mathematica in 1930. The second one appeared in Annals of Mathematics in 1932 and was devoted to Tauberian Theorems. It was precisely with the publication of these two memories that Wiener entered in the "Olympus" of Mathematics, crossing to the first line and obtaining the admiration and recognition of the American mathematicians, a thing he had not got with his previous contributions to Brownian motion, Potential Theory nor Operational Calculus. In 1933 he won the Bôcher Prize and in 1934 he was named Fellow of the National Academy of Sciences of USA.

Although Wiener was admired in life (and after that) by Engineers because of his contributions to Filter Theory, Prediction Theory, Cybernetics, Brownian motion, etc., the mathematicians around the world know him mainly because of his Tauberian Theorems and this unique contribution can be considered enough important to concede him an oustanding place in the History of Mathematical Analysis.

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