Universes in Type-Theoretical Semantics

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This talk

- Brief introduction to MTT-semantics
  - Formal Semantics in Modern Type Theories
- Universes and two examples of linguistic universes
- Subtype universes
  - Bounded quantification, examples, meta-theory

In developing MTT-semantics, I've collaborated with many colleagues, including

- S. Chatzikyriakidis (various respects in MTT-semantics)
- G. Lungu (signatures) and H. Maclean (subtype universes)
- N. Asher (linguistic coercions)
- S. Soloviev, T. Xue and Y. Luo (coercive subtyping)
- R. Adams, P. Callaghan, H. Goguen, R. Pollack (type theory & proof assistants)

I. MTT-semantics

- **Montague Semantics**
  - Montague (1930–1971) & Church's simple TT (1940)
  - Dominating in formal semantics since 1970s

- **Modern Type Theories (MTTs)**
  - Martin-Löf's type theory (predicative);
  - UTT (Luo 1994; impredicative; MTT-semantics so far)
- **MTT-semantics:** formal semantics in modern type theories
  - Ranta (1994): formal semantics in Martin-Löf's type theory
  - Recent development: becoming full-scale alternative to Montague
    - Z. Luo. Formal Semantics in MTTs with Coercive Subtyping. L&P, 35(6). 2012.
    - S. Chatzikyriakidis and Z. Luo. Formal Semantics in MTTs. Wiley/ISTE. 2020.

MTT-semantics: both model/proof-theoretic

- Model-theoretic semantics (traditional)
  - Meaning as denotation (Tarski, …)
  - Montague: NL → (simple TT) → set theory
- Proof-theoretic semantics
  - Meaning as inferential use (Gentzen, …)
  - Not just specified by proof rules, but the rules for proof/consequence must be <u>in harmony</u>.
  - Also: Prawitz, Martin-Löf, Dummett, Brandom (and Wittgenstein)
- MTT-semantics
  - Has both model-theoretic and proof-theoretic characteristics
    - Z. Luo. Formal Semantics in Modern Type Theories: Is It Model-theoretic, Proof-theoretic, or Both? Invited talk at LACL14.
  - In what sense? What does this imply?

- NL → MTT (representational, model-theoretic)
  - MTTs as meaning-carrying languages with <u>types</u> representing collections & <u>signatures</u> representing situations
  - → Powerful tools for wide-range modelling (as in Montague)
- MTT → meaning theory (inferential roles, proof-theoretic)
  - MTT-judgements can be understood proof-theoretically by means of their inferential roles.
  - → Effective NL inference based on proof-theoretic semantics and existing proof technology (Coq, Agda, Lego, …)

Remark: new perspective & new possibility not available before!

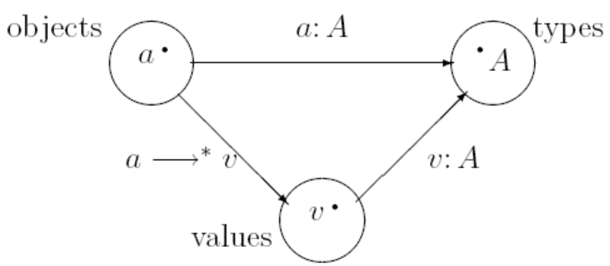
II. Universes

- Example for a first look
  - How to model predicate-modifying adverbs (eg, quickly)?
  - Informally, it can take a verb and return a verb.
- Montague: quickly : $(e \rightarrow t) \rightarrow (e \rightarrow t)$ & quickly(run) : $e \rightarrow t$
- MTT-semantics (where CNs are interpreted as types)?
  - quickly : $(A_{run} \rightarrow Prop) \rightarrow (A_{run} \rightarrow Prop)$, where run : $A_{run} \rightarrow Prop$
  - Other verbs? Adjectives? Generically? One type for all?
- Π-polymorphism comes for the rescue:

  quickly : $\Pi A{:}CN.\ (A \rightarrow Prop) \rightarrow (A \rightarrow Prop)$
- Q: What is CN?

  A: CN is a universe of types that interpret common nouns.

Universes in type theory



- Objects and types:
  - Types collect objects into totalities.
  - Two worlds are connected by "a:A".
- What if we want to collect some types into a totality?
  - E.g., common nouns are types; can we have a type CN whose objects are types that interpret common nouns?
  - Yes, we need a <u>universe</u> CN.
- Notes on Π-polymorphism (e.g., <u>polymorphic</u> quickly)
  - Universes are types and can be quantified over (next page).
  - Note: the collection of all types <u>cannot</u> be quantified over; otherwise, logical paradox.

Universes in linguistic sem: CN as example

- Let's start by reviewing CN
  - quickly : $\Pi A{:}CN.\ (A \rightarrow Prop) \rightarrow (A \rightarrow Prop)$
    - "run quickly" – quickly(A_{run} , run) : $A_{run} \rightarrow Prop$
    - "begin quickly" – quickly(A_{begin} , begin) : $A_{begin} \rightarrow Prop$
- Modelling subsective adjectives
  - Their meanings are dependent on the nouns they modify.
  - Eg, "a large mouse" is not a large animal
  - Our proposal:
    - large : $\Pi A{:}CN.\ (A \rightarrow Prop)$
    - large(Mouse) : Mouse → Prop
    - [large mouse] = $\Sigma x{:}Mouse.\ large(Mouse,x)$

Modelling quantifiers

- Generalised quantifiers
  - Examples: some, most, three, a/an, all, numerals, …
  - In sentences like: "Most students work hard."
- With Π-polymorphism, the type of binary quantifiers is:

$$\Pi A{:}CN.\ (A \rightarrow Prop) \rightarrow Prop$$

For Q of the above type

$$N : CN,\ \ V : N \rightarrow Prop\ \ \Rightarrow\ \ Q(N,V) : Prop$$

E.g., Student : CN, work_hard : Human→Prop

➔ Most(Student, work_hard) : Prop

LType: universe for modelling coordination

- Examples of conjoinable types
  - John walks and Mary talks. (sentences)
  - John walks and talks. (verbs)
  - Mary is pretty and smart. (adjectives)
  - The plant died slowly and agonizingly. (adverbs)
  - Every student and some professors came. (quantified NPs)
  - Some but not all students got an A. (quantifiers)
  - John and Mary went. (proper names)
  - A friend and colleague came. (CNs)
  - … …
- Question: can we consider coordination <u>generically</u>?

- Formal rules of LType in the next slide.
- Then, coordination can be considered generically:
  - Every (binary) coordinator such as <u>And</u> is of type

    $$\Pi A{:}LType.\ A \rightarrow A \rightarrow A$$
- We can then type the coordination examples.
  - Mary is pretty and smart.
    - And(Human→Prop, pretty, smart)(m)
  - Every student and some professors came.
    - And((Human→Prop)→Prop, every(Student), some(Professor))(come)
  - John and Mary went.
    - go(And(Human, j, m))

$$\frac{}{PType : Type} \qquad \frac{}{Prop : PType} \qquad \frac{A : LType \quad P(x) : PType\ [x{:}A]}{\Pi x{:}A.P(x) : PType}$$

$$\frac{}{LType : Type} \qquad \frac{}{\mathrm{CN} : LType} \qquad \frac{A : \mathrm{CN}}{A : LType} \qquad \frac{A : PType}{A : LType}$$

Fig. 1. Some (not all) introduction rules for *LType*.

III. Subtype universes (Maclean & Luo 2021*)

- U(A) is the universe of the subtypes of A.

$$\frac{A\ type}{U(A)\ type} \qquad \frac{B \leq A}{B : U(A)}$$

where ≤ is coercive subtyping (Luo 1997, Luo-Soloviev-Xue 2012).

- Bounded quantification (see, eg, Cardelli-Wegner 85)
  - $\Pi X \leq A. \ldots$ (quantification over the subtypes of a type)
  - Very useful in various constructions (eg, in linguistic sem.)
  - This can be expressed by subtype universe as $\Pi X : U(A). \ldots$
  - BQ is problematic in $F_{\leq}$ (undecidability by Pierce 1994).
    - This has misled/confused myself for a long time!
  - But BQ is OK in our setting (see meta-theory later.)

* H. Maclean and Z. Luo. Subtype Universes. Post-proc. of TYPES20. Leibniz International Proceedings in Informatics, Vol. 188. 2021.

Gradable adjectives: an example

- Gradable adjectives like tall
  - Examples: tall building, tall boy, …
  - Meaning subject to a measure and a threshold
- Let V_h be a universe of CNs whose objects have heights.
  - E.g., Building, Human, … : V_h

Then,

$$\text{height} : \Pi A{:}V_h\ \Pi X{\le}A.\ X \rightarrow \text{Nat} \qquad \text{(measure)}$$

$$\xi : \Pi A{:}V_h\ \Pi X{\le}A.\ \text{Nat} \qquad \text{(threshold)}$$

$$\text{tall} : \Pi A{:}V_h\ \Pi X{\le}A.\ X \rightarrow \text{Prop}$$

$$\text{tall}(A,X,x) = \text{height}(A,X,x) \ge \xi(A,X)$$

E.g., tall(Human,Boy,Oliver), where Boy $\le$ Human.

skilful: another example

- Π-polymorphism for semantics of subsective adjectives
- If skilful : ΠA:CN. (A→Prop)
  - skilful(Doctor) : Doctor → Prop
  - skilful doctor = Σx:Doctor. skilful(Doctor)(x)
- But, could also have "skilful building". How to exclude it?
- skilful : $\Pi A{:}CN_H.\ (A \to Prop)$
  - CN_H – sub-universe of CN (of subtypes of Human)
    - If A : CN and A ≤ Human, then A : CN_H .
  - Then, under the above typing for skilful with CN_H ,
    - skilful(Doctor) : Doctor → Prop because Doctor ≤ Human.
    - skilful(Building) is ill-typed (and excluded) because Building is not a subtype of Human.

Meta-theory of subtype universes

- When adding a universe, one needs to show that the addition is OK.
  - OK in the sense that it preserves nice properties such as logical consistency.
- Adding subtype universes is OK:

Theorem.

> The addition of subtype universes to a type theory preserves its nice properties such as logical consistency and strong normalisation.

See (Maclean & Luo 2021) for a proof.

Thank you!